Math 1031, Self-Evaluation Exercise 1: Solutions
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Name: (*Amy's Solutions*)

Discussion Section: *NA*

Discussion TA: *NA*

This exercise is for your practise. There are four open-ended problems. Give yourself 20 minutes to complete the exercise, and see how you do.

1. Consider $f(x) = 2|x-1| + 3$

   a.) What basic curve can you use to help you graph this function?

   $$y = |x|$$

   b.) Graph the basic curve in (a).

   The graph looks like a "V". It goes through the points $(0,0)$, $(-1,1)$, and $(1,1)$. (Maybe I'll figure out how to insert a picture here??)

   c.) Graph $f(x)$.

   To get this graph from the graph of $y = |x|$, we will do a vertical stretch by a factor of 2, then translate *right* by 1 and *up* by 3. So the three points given above go to:

   $$\begin{aligned} (0,0) &\longrightarrow (1,3) \\ (-1,1) &\longrightarrow (0,5) \\ (1,1) &\longrightarrow (2,5) \end{aligned}$$

   This still has a "V" shape, but it is narrower than the graph of $y = |x|$ and it is shifted. (Picture later?!)

2. If $f(x) = x^2 - 6,\ x \geq 0$

a.) Find the inverse function $f^{-1}(x)$.

We let $y = f(x)$ then switch x and y:

$$\begin{aligned} y &= x^2 - 6, \quad x \geq 0 \\ x &= y^2 - 6, \quad y \geq 0 \end{aligned}$$

Now solve for y:

$$\begin{aligned} x + 6 &= y^2, \quad y \geq 0 \\ y &= \pm\sqrt{x+6}, \quad y \geq 0 \\ y &= \sqrt{x+6} \end{aligned}$$

So $f^{-1}(x) = \sqrt{x+6}$. The domain is $x \geq 6$.

b.) Verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

First, look at $(f \circ f^{-1})(x)$ on the domain of f^{-1}, i.e. for $x \geq 6$:

$$(f \circ f^{-1})(x) = (\sqrt{x+6})^2 - 6 = (x+6) - 6 = x$$

Second, look at $(f^{-1} \circ f)(x)$ on the domain of f, i.e. for $x \geq 0$:

$$(f^{-1} \circ f)(x) = \sqrt{(x^2 - 6) + 6} = \sqrt{x^2} = x$$

c.) Graph $f(x)$ and $f^{-1}(x)$ on the same set of axes.

To graph $f(x) = x^2 - 6,\ x \geq 0$, first think about $y = x^2 - 6$ and then think about the domain restriction $x \geq 0$. Well, $y = x^2 - 6$ is a parabola: take the basic parabola and shift *down* by 6.

$$\begin{aligned} (0,0) &\longrightarrow (0,-6) \\ (1,1) &\longrightarrow (1,-5) \\ (2,4) &\longrightarrow (2,-2) \\ (3,9) &\longrightarrow (3,3) \end{aligned}$$

The domain restriction $x \geq 0$ means that the left half of the parabola is missing.

To graph $f^{-1}(x) = \sqrt{x-6}$, transform the basic square root $y = \sqrt{x}$ by shifting *left* by 6.

$$\begin{aligned} (0,0) &\longrightarrow (-6,0) \\ (1,1) &\longrightarrow (-5,1) \\ (4,2) &\longrightarrow (-2,2) \\ (9,3) &\longrightarrow (3,3) \end{aligned}$$

When you graph both functions on the same set of axes, you will see that the two graphs are symmetric across the line $y = x$.

(Picture later?!)

3. Find two numbers whose sum is 30 and whose product is a maximum.

We want to maximize the *product* in terms of the numbers. Call the product P and the two numbers x and y. We want to write P as a function of x (or as a function of y.) Well,

$$P = xy$$

To eliminate the y, we use the fact that the sum is

$$x + y = 30$$

which means that $y = 30 - x$. So the product is

$$P = P(x) = x(30 - x) = 30x - x^2$$

So P is a quadratic function, and its graph is a parabola pointing downward. That means that P will have a maximum at the vertex,

$$x = -\frac{b}{2a} = -\frac{30}{2(-1)} = 15$$

To finish the problem, we have to make sure we answer the original question: "Find two numbers ..." So we need to find y as well,

$$y = 30 - x = 30 - 15 = 15$$

So the product is a maximum when the two numbers are both equal to 15.

4. Graph $f(x) = -x^3 - x^2 + 6x$.

Put $f(x)$ in factored form:

$$f(x) = -x(x^2 + x + 6) = -x(x - 2)(x + 3)$$

The zeros (x-intercepts) of $f(x)$ are at $x = -3, 0, 2$. This breaks up the number line into four intervals:

$$(-\infty, -3),\ (-3, 0),\ (0, 2),\ (2, \infty)$$

For each interval, we use a test point to see whether $f(x)$ is positive or negative on that interval:

Interval	Test point	Value of f	Pos/neg
$(-\infty, -3)$	$x = -4$	$f(-4) = 4(-6)(-1) = 24$	+
$(-3, 0)$	$x = -1$	$f(-1) = 1(-3)(2) = -6$	-
$(0, 2)$	$x = 1$	$f(1) = (-1)(-1)(4) = 4$	+
$(2, \infty)$	$x = 3$	$f(3) = (-3)(1)(6) = -18$	-

Now graph the cubic using these test points ... (it goes "down, up, down") ...

(Picture later?!)