This is the first part of a two-part lab. Next week (in Lab 6) you will explore the notion of the limit as it applies to the definition of the definite integral. You will write a report on Labs 5-6, which will be due 5pm Thurs Oct 13.

**Goals:**

1. To learn how to generate numerical tables in *Mathematica*.
2. To understand how the notion of a *limit* underlies the definition of continuity and differentiability.
3. To determine continuity and differentiability of functions numerically and graphically.
4. To understand various ways in which continuity or differentiability can fail.

**Groups for the next two weeks:**

| | | | | |
|---|---|---|---|---|
| Yohann<br>Ben<br>Luis | Nathaniel<br>Kelly<br>Shelby | Gretchen<br>Alan<br>Becca | Joel<br>Edgar<br>Justin | Christian A.<br>Garner<br>Christian G. |
| Mitchell<br>Jacob D.<br>Jordan | Aaron<br>Robert<br>Connor | Zoe<br>Grace<br>Jackson | Jacob G<br>Min Ah | |

If the values of $f(x)$ get closer and closer to a finite number as $x$ gets closer and closer to $a$, we call that finite number the *limit* of $f(x)$ as $x$ approaches $a$, and denote it by

$$\lim_{x \to a} f(x)$$

If the values of $f(x)$ do *not* get closer and closer to a finite number as $x$ gets closer and closer to $a$, we say that the limit of $f(x)$ as $x$ approaches $a$ *does not exist.*

1. **Continuity of rational functions**: Recall that function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) \quad = \quad f(a)$$

i.e. the value of the function at $a$ agrees with the limit of $f(x)$ as $x$ approaches $a$. Implicit in this statement are three criteria: (1) the function $f(x)$ must be defined at $a$, (2) the limit of $f(x)$ as $x$ approaches $a$ must exist, and (3) the two must be equal. In this problem, we explore some ways in which continuity can fail, by considering the behavior of three rational functions near $x = 2$.

(a) Consider the function

$$f(x) \quad = \quad \frac{x^2 + x - 6}{x + 2}$$

i. Evaluate $f(x)$ at $x = 2$, if possible. If it is not possible, explain why.

ii. Use the `Table` command to generate lists of five values of $f(x)$, as $x$ approaches 2 from the right and from the left. For example, the *Mathematica* command

```
Table[(x^2 + x - 6)/(x + 2), {x, {3, 2.1, 2.01, 2.001, 2.0001}}]
```

will generate a list of five values of $f(x)$ as $x$ is approaching 2 from the right.

iii. Use your lists of values to make an educated guess for the limit of $f(x)$ as $x$ approaches 2, if it exists. If the limit does not exist, explain why.

iv. Is $f(x)$ continuous at $x = 2$? Why or why not?

v. Plot $f(x)$ near $x = 2$. Does the graph that *Mathematica* generates agree with your conclusions?

(b) Now explore the behavior of the function

$$g(x) \quad = \quad \frac{x^2 + x - 6}{x - 2}$$

near $x = 2$, by completing (i)-(v) but replacing $f(x)$ with $g(x)$.

(c) Finally, explore the behavior of the function

$$h(x) \quad = \quad \frac{x^2 - x - 6}{x - 2}$$

near $x = 2$, by completing the steps (i)-(v) above but replacing $f(x)$ with $h(x)$.

(d) Rewrite the functions $f(x)$, $g(x)$ and $h(x)$ by factoring their numerators. What do you notice? Can you see how you would be able to predict the behavior of these functions near $x = 2$?

2. **Removable discontinuities** A function is said to have a *removable discontinuity* at $x = a$ if the limit of $f(x)$ as $x$ approaches $a$ exists (as a finite number) but $f(x)$ is not defined at $a$. This means that although $f(x)$ is discontinuous at $a$, we can "remove the discontinuity" at $x = a$ by defining a new function:

$$\tilde{f}(x) \quad = \quad \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x \to a} f(x) & \text{if } x = a \end{cases}$$

which *is* continuous at $a$.

(a) Which of the functions in the previous problem has a removable discontinuity at $x = 2$? Define a function that "removes the discontinuity."

(b) The following two functions:

$$f(x) = \sin\left(\tfrac{1}{x}\right) \qquad g(x) = \frac{\sin x}{x}$$

are discontinuous at zero. For each of these two functions, use the the `Table` command to determine the behavior near zero (as in the previous problem) to determine whether the discontinuity at zero is removable or not. If it is removable, define a function that "removes the discontinuity."

3. **Derivatives** Recall that the derivative of a function $f(x)$ at $x = a$, denoted $f'(a)$ is the limit of the average rates of change of $f(x)$ over smaller and smaller intervals containing $a$. More precisely,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In this problem, we will explore the behavior of differentiable and non-differentiable functions by considering the behavior of two functions near $x = 1$.

(a) Consider the function $f(x) = x^2 \sin x$.

i. Use the `Table` command to generate lists of the average rates of change of $f(x)$ over five smaller and smaller intervals containing $x = 1$, from the left and from the right.

ii. Use your lists to make an educated guess for the derivative of $f(x)$ at $x = 1$, if it exists. If the derivative does not exist, explain why.

iii. If the derivative exists, find an equation for the tangent line to the graph of $f(x)$ at $x = 1$, and plot both $f(x)$ and the tangent line on the same set of axes. If the derivative does not exist, plot $f(x)$ near $x = 1$, and explain, in graphical terms, why the derivative does not exist.

(b) Now consider the function $g(x) = \sqrt{|x-1|}$. Complete steps (i)-(iii) above, replacing $f(x)$ with $g(x)$.