# Lab 10: The Alternating Harmonic Series

Names:

## Introduction

In this lab we will explore the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

The terms in this series clearly approach zero, but we know that this fact alone is not sufficient to ensure that the series converges. The example we keep returning to is the harmonic series, whose terms go to zero, but which is divergent.

We will first approach the question numerically, computing partial sums using *Mathematica*. Based on our numerical experiments, we will make a conjecture. Then we will develop what we need to obtain a proof of our conjecture.

## Numerical Experimentation

Using *Mathematica* we will compute partial sums. First we define the sequence of terms:

```
AHTerms[k_] := ((-1)^(k-1))/k;
```

Next we define the sequence of partial sums:

```
AHSums[n_] := Sum[AHTerms[k], {k, 1, n}];
```

The following table will show the first 10 partial sums:

```
TableForm[N[Table[{k, AHSums[k]}, {k, 1, 10}]]]
```

Modify this command to compute higher and higher partial sums. (Change the lower limit for $k$ as well as the upper limit; otherwise you will get very large tables.) Do you think that the alternating harmonic series converges? If so, give a decimal approximation for the limit, which you believe to be accurate up to 3 decimal places.

If you look at every other partial sum, what do you notice?

In order to prove our conjecture, we will look at every other partial sum. In other words, we will examine two sequences of partial sums separately, the sequence of partial sums with even index, $S_{2n}$, and the sequence of partial sums with odd index, $S_{2n+1}$.

# Two Warm-ups

## ■ Looking at every other partial sum (14.4.6)

Consider the infinite series $a_0 + a_1 + \ldots$, where

$$a_k = \left(\frac{2}{3}\right)^k + (-1)^k \cdot (0.0001)$$

We will use *Mathematica* to compute partial sums of even index and partial sums of odd index. First we define $a_k$, as follows:

```
a[k_] := (2 / 3) ^k + ((-1) ^k) * (0.0001);
```

Next we define the partial sums:

```
S[n_] := Sum[a[k], {k, 0, n}];
```

And finally, we create a table to show the partial sums $S_n$ from $n = 0$ to $n = 10$.

```
TableForm[Table[{n, S[n]}, {n, 0, 10}]]
```

Modify this command to compute higher and higher partial sums. (As before, you should change the lower limit for $k$ as well as the upper limit; otherwise you will get very large tables.) Fill in the blanks below with some conjectures:

$$S_{2n} \longrightarrow \qquad \text{as } 2n \longrightarrow \infty$$

$$S_{2n+1} \longrightarrow \qquad \text{as } 2n+1 \longrightarrow \infty$$

We can prove our conjectures using what we know about geometric series. Let

$$b_k = \left(\frac{2}{3}\right)^k$$
$$c_k = (-1)^k \cdot (0.0001)$$

Then we can rewrite our original sequence of terms as

$$a_k = b_k + c_k$$

And the partial sums $S_n$ can be written in terms of partial sums for the two series $b_0 + b_1 + \ldots$ and $c_0 + c_1 + \ldots$, which can be easily obtained. What are they?

$$b_0 + b_1 + \ldots + b_n =$$

$$c_0 + c_1 + \ldots + c_n = \qquad \text{(if } n \text{ is even)}$$

$$c_0 + c_1 + \ldots + c_n = \qquad \text{(if } n \text{ is odd)}$$

What can we conclude about the limits of the sequences $S_{2n}$, $S_{2n+1}$, and $S_n$, as $n \to \infty$?

What can you say about the limits of the sequences $a_{2n}$, $a_{2n+1}$, and $a_n$, as $n \to \infty$?

## The Euler-Mascheroni constant (14.5.1)

Remember that in Exercise 7.1.9, we proved that the sequence

$$a_n = \ln(n) - \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

is increasing and bounded (in particular, $0 < a_n < 1$). What does the Monotone Bounded Convergence Theorem allow us to conclude about the sequence $(a_n)$?

Now consider the related sequence

$$b_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln(n)$$

How are $a_n$ and $b_n$ related? (Write an equation.)

The limit of the sequence $(b_n)$ is called the Euler-Mascheroni constant and denoted $\gamma$. Explain how we know that $(b_n)$ converges, and write $\gamma$ in terms of the limit of $(a_n)$.

# The limit of the alternating harmonic series (14.5.3)

First we look at the partial sums of even index. We will show that they are related to the partial sums of the harmonic series.

By experimenting by hand, with small values of $n$, find a relationship between the following partial sums:

$$A_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n}$$
$$H_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots - \frac{1}{2n}$$
$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots - \frac{1}{n}$$

Write this relationship below:

In order to use the results of the preceding section (on the Euler-Mascheroni constant), we will next cleverly can add and subtract $\ln(2n)$, to obtain a relationship between the following:

- $A_{2n}$
- $H_{2n} - \ln(2n)$
- $H_n - \ln(n)$
- $\ln(2)$

Write this relationship below:

Using this relationship, as well as the results of the preceding section, find the limit of $A_{2n}$ as $n$ approaches infinity.

Next we will look at the partial sums of odd index. Write $A_{2n+1}$ in terms of $A_{2n}$ and the $(2n+1)^{\text{th}}$ term, $a_{2n+1}$, of the alternating harmonic series.

Use this to find the limit of $A_{2n+1}$ as $n$ approaches infinity.

The alternating series converges if the sequence $A_n$ of partial sums converges. We have shown that the subsequence $A_{2n}$ converges and that the subsequence $A_{2n+1}$ also converges. What can we conclude about the sequence $A_n$? Explain.

State your conclusion about the limit of the alternating harmonic series as a complete mathematical sentence.

Does this conclusion agree with the conjecture you made at the beginning of lab?