Consider the matrices A and B below. The matrices R_A and R_B are the reduced row echelon forms of A and B respectively.

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{pmatrix} \qquad R_A = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & 2 \end{pmatrix} \qquad R_B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The purpose of the exercise is to determine what information R_A (and R_B) tell us about the subspaces associated with A (resp. B).

1. Recall that the elementary row operations do not change the span of the row vectors of a matrix.
   (a) What does this imply about the row space of A and the row space of R_A (or the row space of B and the row space of R_B)?
   (b) Use this information to find a basis for the row space of A and the row space of B.

2. Recall that elementary row operations do not change the dependency relations among the columns of a matrix. (That is why we can use the reduced row echelon form to determine dependency relations among column vectors.)
   (a) Look at the column vectors of R_A and choose a basis for the column space of R_A from among those columns.
   (b) Use this information to determine a basis for the column space of A.
   (c) Use the same strategy to determine a basis for the column space of B.

3. Recall that the null space of a matrix M is the set of all vectors v such that $Mv = 0$. Applying elementary row operations to the augmented matrix $(M|0)$ is equivalent to multiplying both sides by elementary matrices.
   (a) What does this imply about the null space of a matrix and the null space of its reduced row echelon form?
   (b) Describe the vectors in the null space of R_A entry-wise. Determine a basis for the null space of R_A and the null space of A.
   (c) Use the same strategy to determine a basis for the null space of B.

4. What are the dimensions of the row spaces, column spaces, and null spaces of A and B? What pattern(s) do you notice? Will these patterns hold in general? Why or why not?