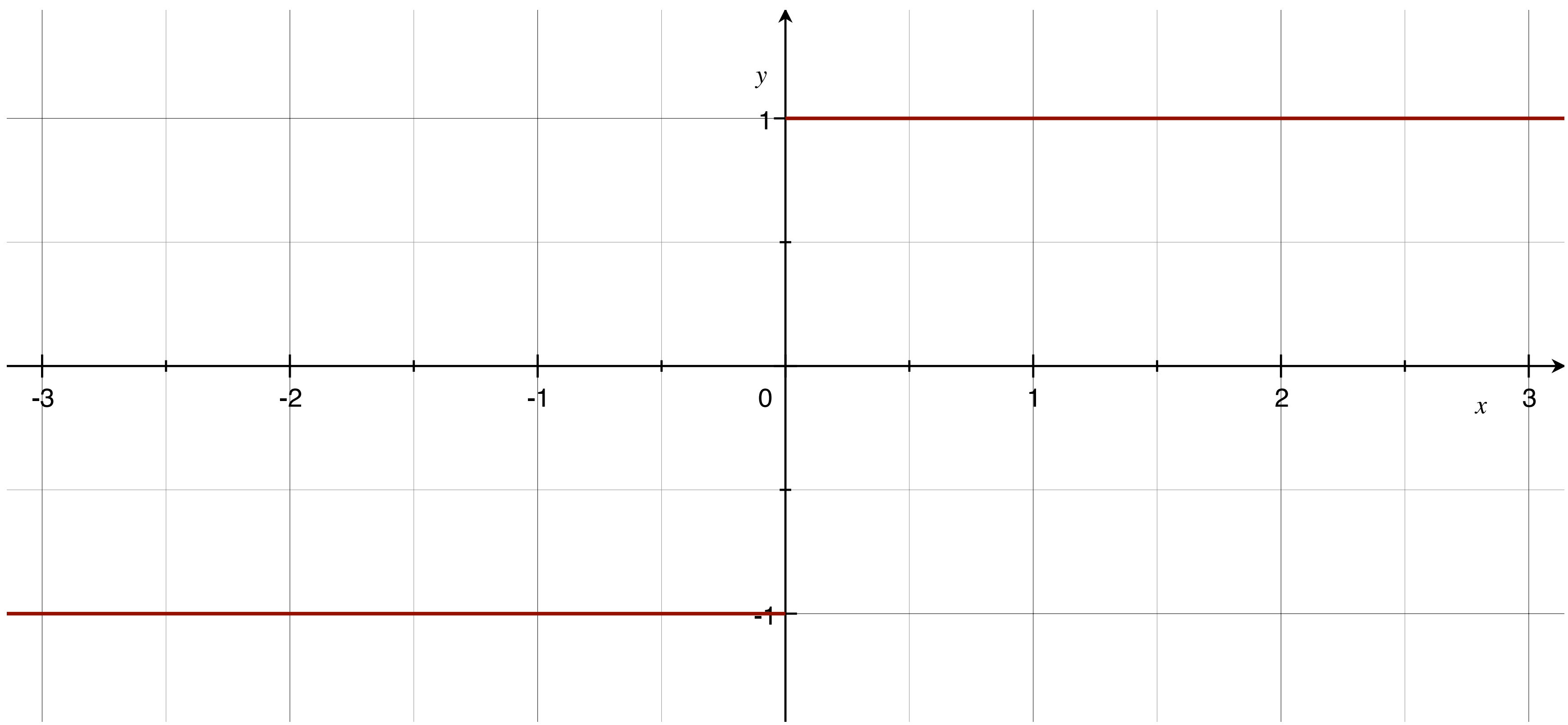
Let $f(x)$ be the step function on $[-\pi, \pi]$:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



We have shown that the set V of bounded functions on the closed interval $[-\pi, \pi]$ that are piecewise continuous is a vectorspace, with an inner product given by

$$\langle f, g \rangle \;=\; \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

Further, with respect to this inner product, the functions

$$1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x\ ,\ldots,\ \sin kx,\ \cos kx, \ldots$$

are mutually orthogonal in V, and

$$\langle \sin kx, \sin kx \rangle \;=\; \pi \qquad \langle \cos kx, \cos kx \rangle \;=\; \pi \qquad \langle 1, 1 \rangle \;=\; 2\pi$$

This orthogonal set of vectors (functions) *spans* V in the sense that any vector (function) in V can be approximated arbitrarily well by linear combinations of vectors in this orthogonal set.

Let W_0 be the subspace of V spanned by 1, W_1 be the subspace spanned by 1, $\sin x$, and $\cos x$, and so on:

$$\begin{aligned}
W_0 &= \operatorname{span}\big(\{1\}\big) \\
W_1 &= \operatorname{span}\big(\{1, \sin x, \cos x\}\big) \\
W_k &= \operatorname{span}\big(\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos kx\}\big)
\end{aligned}$$

The zeroth approximation is:

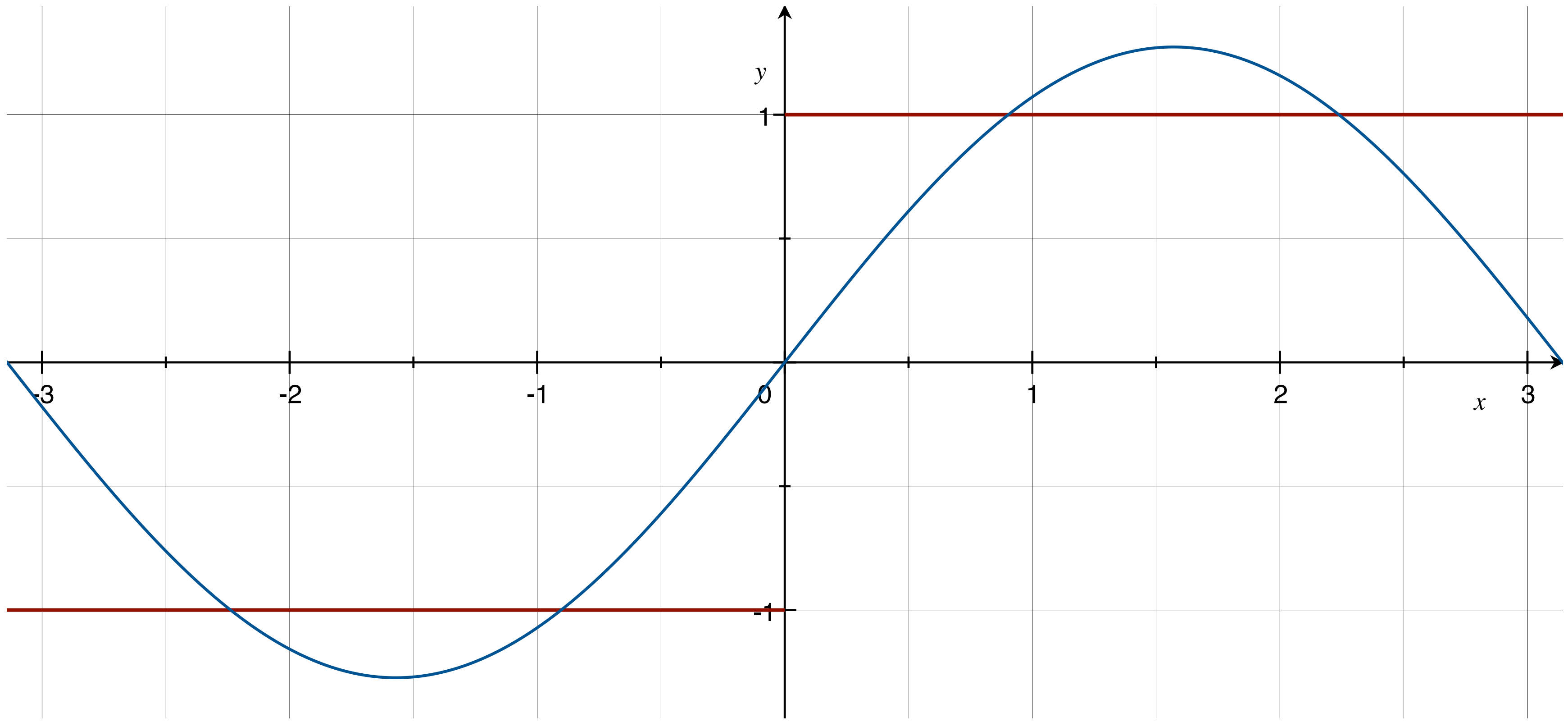
$$f_0 \;=\; \mathrm{Proj}_{W_0} f \;=\; \mathrm{Proj}_1 f \;=\; \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}\, 1 \;=\; \frac{0}{2\pi}\, 1 \;=\; 0$$

The first approximation is:

$$\begin{aligned}
f_1 &= \mathrm{Proj}_{W_1} f \\
&= \mathrm{Proj}_1 f \;+\; \mathrm{Proj}_{\cos x} f \;+\; \mathrm{Proj}_{\sin x} f \\
&= \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \cdot 1 \;+\; \frac{\langle f, \cos x\rangle}{\langle \cos x, \cos x\rangle} \cdot \cos x \;+\; \frac{\langle f, \sin x\rangle}{\langle \sin x, \sin x\rangle} \cdot \sin x \\
&= 0 \;+\; 0 \;+\; \frac{4}{\pi}\, \sin x
\end{aligned}$$

The first approximation is

$$f_1(x) \quad = \quad \frac{4}{\pi} \sin x$$



In general, the Nth approximation is:

$$f_N \;=\; \mathrm{Proj}_{W_N} f \;=\; \mathrm{Proj}_1 f \;+\; \sum_{k=1}^{N} \left(\mathrm{Proj}_{\sin kx} f + \mathrm{Proj}_{\cos kx} f\right)$$

For the step function,

$$\mathrm{Proj}_1 f \;=\; 0$$

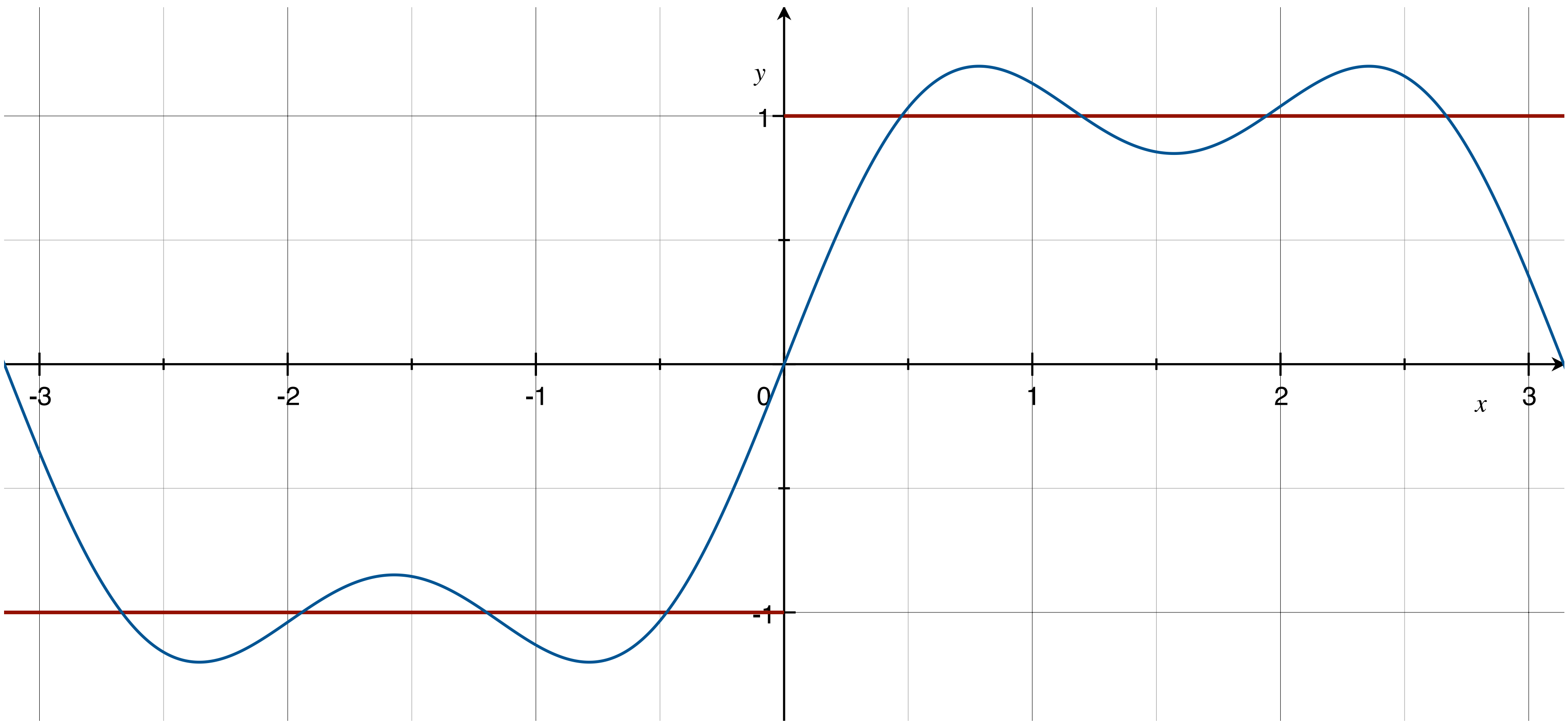
$$\mathrm{Proj}_{\cos kx} \;=\; 0 \qquad (\text{for all } k > 0)$$

$$\mathrm{Proj}_{\sin kx} \;=\; 0 \qquad (\text{for even } k > 0)$$

$$\mathrm{Proj}_{\sin kx} \;=\; \frac{4}{\pi k}\,\sin kx \qquad (\text{for odd } k > 0)$$

So the second approximation is the same as the first, but the third approximation is

$$f_3 = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin(3x)$$



Writing odd k as $k = 2n + 1$, we can see that the Fourier series for the step function is:

$$\sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\big((2n+1)x\big)$$

This movie shows the first twenty approximations.