Let V be the set of bounded functions on the closed interval $[-\pi, \pi]$ that are piecewise continuous (i.e. continuous except perhaps at finitely many points.)

1. Show that V is a (real) vectorspace.

2. Show that

$$\langle f, g \rangle \;\; = \;\; \int_{-\pi}^{\pi} f(x)\, g(x)\; dx$$

is an inner product on V.

3. Show that the functions

$$1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x\ ,\ldots,\ \sin kx,\ \cos kx,\ldots$$

are mutually orthogonal in V; i.e. show that

(a) $\langle 1, \sin kx\rangle \ =\ 0$ for all positive integers k

(b) $\langle 1, \cos kx\rangle \ =\ 0$ for all positive integers k

(c) $\langle \sin kx, \sin \ell x\rangle \ =\ 0$ for all positive integers $k \neq \ell$

(d) $\langle \cos kx, \cos \ell x\rangle \ =\ 0$ for all positive integers $k \neq \ell$

(e) $\langle \sin kx, \cos \ell x\rangle \ =\ 0$ for all positive integers k and ℓ

For (c), (d), (e), you may find the product-to-sum formulas helpful:

$$\begin{aligned}
\sin\theta\sin\varphi &= \frac{\cos(\theta-\varphi)\ -\ \cos(\theta+\varphi)}{2}\\
\cos\theta\cos\varphi &= \frac{\cos(\theta-\varphi)\ +\ \cos(\theta+\varphi)}{2}\\
\sin\theta\cos\varphi &= \frac{\sin(\theta+\varphi)\ +\ \sin(\theta-\varphi)}{2}
\end{aligned}$$

4. Show that

   (a) $\langle \sin kx, \sin kx \rangle \;=\; \pi$

   (b) $\langle \cos kx, \cos kx \rangle \;=\; \pi$

   (c) $\langle 1, 1 \rangle \;=\; 2\pi$