The purpose of this activity is to classify the isometries of the plane. Consider the various linear transformations of the plane that we have discussed: dilations, rotations, horizontal/vertical expansions/compressions, horizontal/vertical shears, reflections, projections. Which are isometries? Could there be other linear transformations (that we have not discussed) that are isometries of the plane?

1. Matrix representations for rotations and reflections.

   (a) Consider the linear transformation R_θ which is a counter-clockwise rotation by θ. Draw and label $R_\theta \hat{\imath}$ and $R_\theta \hat{\jmath}$. What is the standard matrix representation for R_θ?

   (b) Consider the linear transformation S_ϕ which is a reflection across the line through the origin that makes an angle ϕ with the x-axis. Draw and label $S_\phi \hat{\imath}$ and $S_\phi \hat{\jmath}$. What is the standard matrix representation for S_ϕ?

2. Any isometry T of the plane will map $\hat{\imath}$ to a unit vector, which can be written in the form $u_1 = (\cos\alpha, \sin\alpha)$. The image of $\hat{\jmath}$ will be a unit vector $u_2 = (\cos\beta, \sin\beta)$ that is orthogonal to u_1.

   (a) What are the possible values of β, in terms of the given α? (Hint: Use the trig. identity $\cos x \cos y + \sin x \sin y = \cos(x - y)$.)

(b) Using (a), determine the possible vectors for u_2 in terms of α. For each possible u_2, draw u_1 and u_2 on the same set of axes. Do your answers make sense? (Hint: $\cos(x \mp \frac{\pi}{2}) = \pm \sin(x)$ and $\sin(x \pm \frac{\pi}{2}) = \pm \cos(x)$.)

(c) Using (b), write the possible matrix representations for isometries of the plane.

(d) Compute the determinants of the matrices in (c).

3. Compare your findings from (1) and (2). What can you conclude?