

A linear transformation is a particular kind of function that takes *vectors* as inputs and outputs. We will take some time to study linear transformations of the plane in order to develop intuition for some of the things we will learn later in the semester.

## 1. Functions

First we recall some things about functions. A function takes inputs and gives outputs in a consistent manner, i.e. given a particular input, it always gives the same output.

$$\text{input} \longrightarrow \boxed{\text{function}} \longrightarrow \text{output}$$

You are probably most familiar with functions whose inputs and outputs are real numbers. For example, we usually write  $f(x) = x^2$  to denote the function that takes a real number as an input and gives the square of that number as the output. But the term *function* applies to any assignment of outputs to inputs in a consistent manner. In particular, we will soon consider functions which take vectors in the plane as inputs and outputs.

Recall that the *domain* of a function is the set of allowed inputs and the *range* is the set of all outputs. For example the domain of  $f(x) = x^2$  is the set of all real numbers (unless we are working under some kind of externally imposed constraint), and the range is the set of all non-negative real numbers. The output assigned to a given input  $x$  is sometimes called the *image* of  $x$ , and for this reason the range of a function  $f$  is sometimes also called the *image* of  $f$ .

A function has an *inverse* if and only if it is one-to-one, i.e. every output comes from a unique input. In that case, the domain of the inverse function is the range of the original function.

## 2. Linear transformations

Can you think of any systematic ways of moving vectors around in the plane (keeping the origin fixed, of course)? What about rotating the whole plane by  $90^\circ$ ? Or reflecting across the  $y$ -axis? Or dialating by a factor of 3? All of these are examples of *linear transformations*.

A *linear transformation* of the plane is a function  $T$  that takes vectors in the plane as inputs and outputs such that:

- (1)  $T(\alpha v) = \alpha T(v)$  for all scalars  $\alpha$  and all vectors  $v$
- (2)  $T(v + w) = T(v) + T(w)$  for all vectors  $v$  and  $w$

The first property means that if you first dialate, then apply  $T$ , you get the same result as if you applied  $T$  first then dialated. The second property means that  $T$  takes triangles to triangles.

Note that these two properties imply that  $T(0)$  is *always* the zero vector. (*Exercise: prove this.*) The set of *all* vectors whose image is the zero vector is called the *kernel* of  $T$ :

$$\ker T = \{\text{vectors } v \text{ such that } T(v) = 0\}$$

The fact that  $T(0)$  is always zero implies that the zero vector is always in the kernel.

Some examples of linear transformations are: dialations, rotations about the origin, reflections across lines through the origin, horizontal or vertical contractions or expansions, horizontal or vertical shears, and projections. (*Exercise: Convince yourself that these really do satisfy the defining properties of a linear transformation.*)

## 3. Matrices

Notice that if  $T$  is a linear transformation, then knowing  $T(\hat{i})$  and  $T(\hat{j})$  is sufficient to determine  $T$  uniquely. (*Exercise: prove this.*)

A *matrix* is a common notational device for a linear transformation; it is an array of numbers whose columns are the vectors  $T(\hat{i})$  and  $T(\hat{j})$ . For example, if  $T$  is the  $90^\circ$  counter-clockwise rotation, we write

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Some books make a big distinction between a linear transformation and the matrix that represents the linear transformation. But really, the matrix is just a way of writing out the linear transformation in coordinates.

To compute the action of a linear transformation on a vector in coordinates:

$$Tv = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{1,1}v_1 + a_{1,2}v_2 \\ a_{2,1}v_1 + a_{2,2}v_2 \end{pmatrix}$$

(Exercise: Why does this work?)

#### 4. Eigenvectors and eigenvalues

An *eigenvector*  $v$  of a linear transformation  $T$  is a vector whose image under  $T$  is just a scalar multiple of itself, i.e.

$$Tv = \lambda v \quad \text{for some scalar } \lambda$$

That scalar  $\lambda$  is called the *eigenvalue*.

For example, if  $T$  is the reflection across the  $x$ -axis, then

$$T(\hat{i}) = -\hat{i} \quad \text{and} \quad T(\hat{j}) = \hat{j}$$

so  $\hat{i}$  and  $\hat{j}$  are both eigenvectors for  $T$ , with eigenvalues  $-1$  and  $+1$ , respectively. Notice that

$$T(5\hat{i}) = -5\hat{i} \quad \text{and} \quad T(17\hat{j}) = 17\hat{j} \quad \text{etc.}$$

i.e.  $5\hat{i}$  is another eigenvector for  $T$  with eigenvalue  $-1$ , and  $17\hat{j}$  is another eigenvector with eigenvalue  $+1$ . In fact, any scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue. (Exercise: prove this).