p 524 Ch 7 Test
5-28

p 582 Ch 8 Test
1-20

Math 109, Review for Exam 2

Name: Amy **Section:** ______

Instructions: The exam will have nine problems. Make sure to show all your work and make your final answer clear. Include labels and units when appropriate. No notes, books, or calculators are permitted during the exam.

1. Fill in the blanks.

   (a) For $x > 0$, $y = 3^x$ if and only if x = $\log_3 y$

   (b) After t years, the balance A in an account with principal P an annual interest rate r (in decimal form) is given by the following formulas:

   i. For n compoundings per year: A = $P(1+r/n)^{n \cdot t}$

   ii. For continuous compounding: A = Pe^{rt}

   (c) The x-intercept of the function $\log_2 x$ is x = 1.

   (d) The range of the function $e^x + 2$ is $(2, \infty)$.

   (e) The domain of the function $\log(x-2)$ is $(2, \infty)$.

   (f) Suppose n is any real number and u and v are positive real numbers. Decide whether the following statements are true or false.

   i. $\ln(u) \cdot \ln(v) \stackrel{?}{=} \ln(u) + \ln(v)$ F

   ii. $\ln(u) + \ln(v) \stackrel{?}{=} \ln(u+v)$ F

   $\ln(u \cdot v) = \ln u + \ln v$

   iii. $\ln(u) - \ln(v) \stackrel{?}{=} \ln(u/v)$ T

   iv. $\ln(u/v) \stackrel{?}{=} \frac{\ln(u)}{\ln(v)}$ F

   v. $(\ln u)^{1/2} \stackrel{?}{=} \frac{1}{2}\ln u$ F

   vi. $\ln\sqrt{u} \stackrel{?}{=} \frac{1}{2}\ln u$ T

   $\ln(u^n) = n \ln u$

   vii. $\ln(1/u) \stackrel{?}{=} -\ln(u)$ T $\ln(1/u) = \ln(1) - \ln(u) = -\ln u$

   (g) The antiderivative of the function 6^x is $\left(\frac{1}{\ln 6}\right) 6^x$.

2. Evaluate the function at the given value of x.

   (a) $f(x) = \log x; \quad x = 1000$

   $f(1000) = \log(1000) = 3 \qquad (\text{since } 10^3 = 1000)$

   (b) $g(x) = \log_9 x; \quad x = 3$

   $g(3) = \log_9(3) = 1/2 \qquad (\text{since } 9^{1/2} = 3)$

   (c) $h(x) = \log_2 x; \quad x = 1/4$

   $h(1/4) = \log_2(1/4) = -2 \qquad (\text{since } 2^{-2} = \frac{1}{2^2} = \frac{1}{4})$

   (d) $p(x) = \log_3 x; \quad x = 1/81$

   $p(1/81) = \log_3(81) = -4 \qquad (\text{since } 3^{-4} = \frac{1}{3^4} = \frac{1}{81})$

3. For each of the following functions, find the domain, range, intercepts, and asymptotes, and sketch a graph, labeling at least three points.

   (a) $f(x) = 3^x + 1$

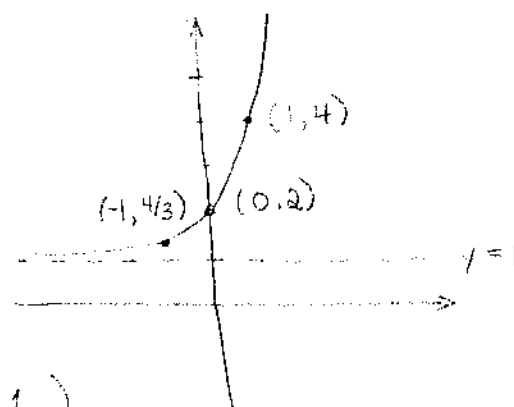
   Domain: $(-\infty, \infty)$

   Range: $(1, \infty)$

   y-intercept: $y = 2$

   Horiz Asymp: $y = 1$

   (Shift graph of 3^x up by 1.)



   (b) $g(x) = \log_3(x-1)$

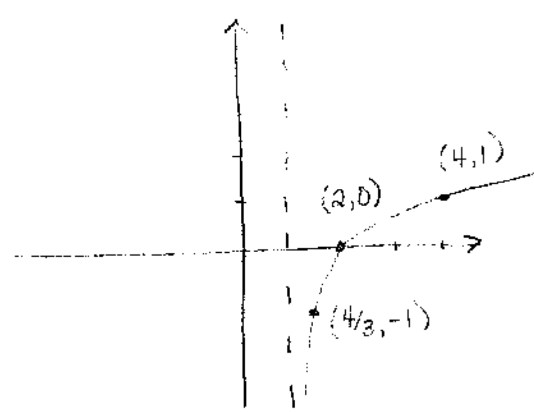
   Domain: $(1, \infty)$

   Range: $(-\infty, \infty)$

   x-intercept: $x = 2$

   Vert. Asymp. $x = 1$

   (Shift graph of $\log_3 x$ right by 1.)



4. Solve for x.

(a) $2^x - 3 = 29$

$2^x = 32$

$\boxed{x = 5}$

(b) $\log_8(x-1) = \log_8(x-2) - \log_8(x+2)$

$\log_8(x-1) = \log_8\left(\frac{x-2}{x+2}\right)$

$x-1 = \frac{x-2}{x+2}$

$(x-1)(x+2) = x-2$

$x^2 + 2x - x - 2 = x - 2$

$x^2 + x - 2 = x - 2$

$x^2 + x = x$

$x^2 = 0$

$x = 0$

But $\log_8(x-1)$ needs $x > 1$

$\log_8(x-2)$ needs $x > 2$

$\log_8(x+2)$ needs $x > -2$

Therefore $x = 0$ is an extraneous solution.

The equation has no real solutions

5. Differentiate.

(a) $\frac{d}{dt}(2^t + t^2) = (\ln 2)\, 2^t + 2t$

(b) $\frac{d}{dx}\ln(4x) = \frac{1}{4x} \cdot 4 = \boxed{\frac{1}{x}}$

(Also: $\ln(4x) = \ln(4) + \ln(x)$
constant
So $\frac{d}{dx}\ln 4x = \frac{d}{dx}(\ln 4 + \ln x) = 0 + \frac{1}{x} = \frac{1}{x}$ ✓)

(c) $\frac{d}{dx}\underbrace{e^{-x}(e^x + e^{-x})}_{1 + e^{-2x}} = 0 + e^{-2x}(-2) = \boxed{-2e^{-2x}}$

(d) $\frac{d}{dt}(\log_3 t)^8 = 8(\log_3 t)^7 \cdot \frac{d}{dt}(\log_3 t)$

$= 8(\log_3 t)^7 \cdot \frac{1}{\ln 3 \cdot t} = \left(\frac{8}{\ln 3}\right)\frac{(\log_3 t)^7}{t}$

Math 109, Review for Exam 2

(e) $\frac{d}{du}\, e^{2u} \cdot \ln(u)$ $= (e^{2u} \cdot 2)(\ln u) + (e^{2u}) \cdot \left(\frac{1}{u}\right)$

$= 2e^{2u} \ln u + e^{2u}\left(\frac{1}{u}\right)$

$= \boxed{\left(2 \ln u + \frac{1}{u}\right) e^{2u}}$

(f) $\dfrac{d}{dv}\, \dfrac{7v-1}{\log(v)}$ $= \dfrac{7(\log v) - (7v-1)\left(\frac{1}{\ln(10)\, v}\right)}{(\log v)^2}$

$= \boxed{\dfrac{7}{\log v} - \dfrac{7v-1}{(\ln 10)\, v\, (\log v)^2}}$

6. Evaluate the indefinite integrals.

(a) $\int e^t + e^3\, dt$ $= e^t + (e^3)\, t + C$

(b) $\int 3x^2 e^{x^3}\, dx$ $= \int e^u\, du = e^u + C = e^{x^3} + C$

$u = x^3$
$du = 3x^2\, dx$

(c) $\displaystyle\int \frac{2}{3-7x}\, dx$ $= \displaystyle\int \frac{2}{3-7x}\left(-\frac{1}{7}\right)(-7\, dx) = -\frac{2}{7}\int \frac{1}{u}\, du = -\frac{2}{7}\ln|u| + C$

$u = 3-7x$
$du = -7\, dx$

$= \boxed{-\frac{2}{7}\ln|3-7x| + C}$

(d) $\displaystyle\int \frac{4x+10}{x^2+5x}\, dx$ $= \displaystyle\int \frac{2(2x+5)}{x^2+5x}\, dx = \int \frac{2}{u}\, du = 2\ln|u| + C$

$u = x^2+5x$
$du = (2x+5)\, dx$

$= \boxed{2\ln|x^2+5x| + C}$

7. Find the area of the region bounded by the graphs of the equations:

$$y = xe^{-x^2} \qquad y = 0 \qquad x = 0 \qquad x = 4$$

$$A = \int_0^4 xe^{-x^2}\,dx \qquad u = -x^2 \qquad u(0) = 0$$

$$du = -2x\,dx \qquad u(4) = -16$$

$$= \int_0^4 e^{-x^2}\left(-\tfrac{1}{2}\right)(-2x\,dx)$$

$$= -\frac{1}{2}\int_{-16}^{0} e^{u}\,du$$

$$= -\frac{1}{2}\left(e^{u}\Big|_{-16}^{0}\right)$$

$$= -\frac{1}{2}\left(e^{0} - e^{-16}\right) = -\frac{1}{2}\left(1 - e^{-16}\right) = \boxed{\frac{1}{2}\left(e^{-16} - 1\right)}$$

8. For $y = (x^2-1)^{2x+1}$, find $\frac{dy}{dx}$ using logarithmic differentiation.

$$y = (x^2-1)^{2x+1}$$

$$\ln(y) = \ln\left((x^2-1)^{2x+1}\right)$$

$$= (2x+1)\ln(x^2-1)$$

$$= (2x+1)\ln\left((x-1)(x+1)\right)$$

$$= (2x+1)\left(\ln(x-1) + \ln(x+1)\right)$$

$$\frac{1}{x-1} + \frac{1}{x+1} = \frac{(x-1)+(x+1)}{(x-1)(x+1)} = \frac{2x}{x^2-1}$$

$$\frac{y'}{y} = 2\left(\ln(x-1) + \ln(x-1)\right) + (2x+1)\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

$$= 2\ln(x^2-1) + \frac{2x(2x+1)}{x^2-1}$$

$$\boxed{y' = (x^2-1)^{2x+1}\left(2\ln(x^2-1) + \frac{2x(2x+1)}{x^2-1}\right)}$$

9. Find the relative extrema and inflection points of $f(x) = \ln x - x$. (Make sure to include x and y-coordinates for each local maximum, local minimum, and inflection point.)

$$f'(x) = \frac{1}{x} - 1$$

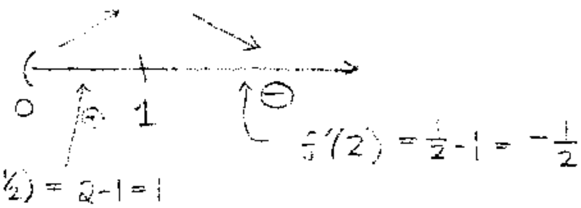
NB: Domain of $f(x)$ is $(0, \infty)$.

$$f''(x) = -\frac{1}{x^2}$$

- $f'(x) = 0$ when $\frac{1}{x} = 1$ i.e. $x = 1$.

  $f'(x)$ DNE when $x = 0$ (not in domain)

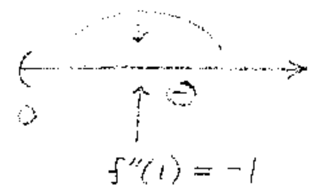
  $\Rightarrow$ Rel. max at $x = 1$. $f(1) = 0 - 1 = -1$



- $f''(x) = 0$ when $\frac{-1}{x^2} = 0$ (impossible)

  $f''(x)$ DNE when $x = 0$ (not in domain)

  $\Rightarrow$ No infl pts.



Rel. max at $x = 1$. Max val. is $f(1) = -1$

10. After t years, the value (in thousands of dollars) of a car that originally cost \$24,000 is given by $V(t) = 24\,(3/4)^t$. How many years will it take for the car to depreciate to half its initial value? (Leave your answer in exact form.)

$$12 = 24\,(3/4)^t$$

$$\frac{12}{24} = \left(\frac{3}{4}\right)^t$$

$$\frac{1}{2} = \left(\frac{3}{4}\right)^t$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right)^t$$

$$\ln\left(\frac{1}{2}\right) = t\ln\left(\frac{3}{4}\right)$$

$$\Rightarrow t = \frac{\ln(1/2)}{\ln(3/4)}$$

It will take $\frac{\ln(1/2)}{\ln(3/4)}$ years to depreciate to half its value.