Name: ____

Names of collaborators: ____

Main Points to Review:

1. One-to-one functions, inverse functions
2. Logarithms, inverse trig functions
3. Composition of functions

1. Inverse Functions

Recall that a function is a relationship between two quantities, in which one quantity (e.g. position within this room) uniquely determines the other (e.g. air temperature). Notice that, in this case, the inverse relationship *cannot* be modeled with a function: the air temperature does not uniquely determine a position in the room. In contrast, consider the example of a population that constantly increases over time; in this case, given a population size, say 10 billion people, we could say exactly when that size was attained, e.g. 1971. When a given output ($f(x)$-value) uniquely determines an input (x-value), then the function is *one-to-one*.

Every one-to-one function has an inverse function, i.e. a function that switches inputs and outputs. (You can tell that a function is one-to-one if it passes the horizontal line test.) Inverses *undo* eachother, i.e. if we input x to a function f, and get y as an output, then we input y into the inverse function f^{-1} of f, then we get x back as the output; we're back where we started. In other words: if $f(x) = y$, then $f^{-1}(y) = x$.

The graph of f^{-1} is obtained by reflecting the graph of f through the line $y = x$. (You're basically just switching the role of x and y.)

Exercise:

1. Determine whether each function is one-to-one. If it is one-to-one, sketch a graph of its inverse on the same set of axes.

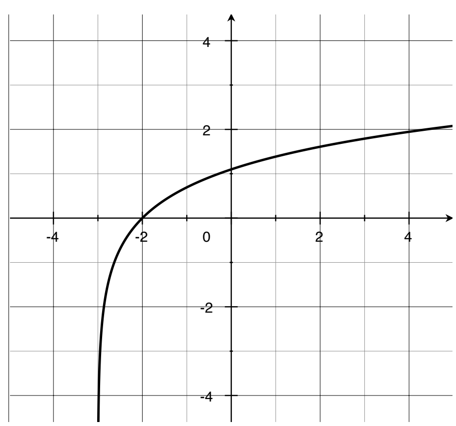


Figure 1: Is this function one-to-one? Y/N

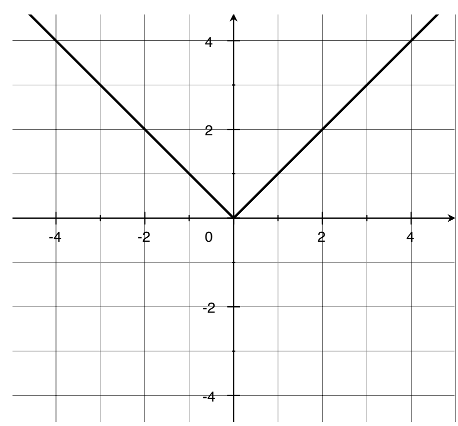


Figure 2: Is this function one-to-one? Y/N

3. Logarithms and Inverse Trig Functions

Logarithms are the inverse functions for exponential functions. For a number $b \neq 0$, we define the logarithmic function with base b by:

$$\log_b(x) = y \quad \Longleftrightarrow \quad b^y = x.$$

The natural logarithm, denoted ln, is the logarithm base e. So,

$$\ln(x) = y \quad \Longleftrightarrow \quad e^y = x.$$

Notice that since the exponential functions always have positive outputs, the logarithms only make sense for positive inputs. Thus the domain of the function $f(x) = \log_b(x)$ is $(0, \infty)$.

Laws of logarithms: For $b \neq 0$, x, y positive real numbers, and r any real number,

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
- $\log_b(x^r) = r \log_b(x)$

Another fact that sometimes comes in handy is the change of base formula:

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Since the trig. functions are not one-to-one, they do not have proper inverses. It is necessary to restrict the domain of the trig functions in order to find inverse trig. functions. For example, we restrict sine to the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The inverse of this function is the inverse sine function, denoted $\sin^{-1} x$, also called the arcsine function, denoted $\arcsin x$.

$$\begin{array}{rclcrclcrcccl}
\arcsin(x) & = & y & \Longleftrightarrow & \sin(y) & = & x & \text{and} & -\frac{\pi}{2} & \leq & x & \leq & \frac{\pi}{2} \\
\arccos(x) & = & y & \Longleftrightarrow & \cos(y) & = & x & \text{and} & 0 & \leq & x & \leq & \pi \\
\arctan(x) & = & y & \Longleftrightarrow & \tan(y) & = & x & \text{and} & -\frac{\pi}{2} & < & x & < & \frac{\pi}{2}
\end{array}$$

Exercises:

2. Without using a calculator, find the exact value of

   (a) $\ln(1/e)$

   (b) $\log_2 6 - \log_2 15 + \log_2 20$

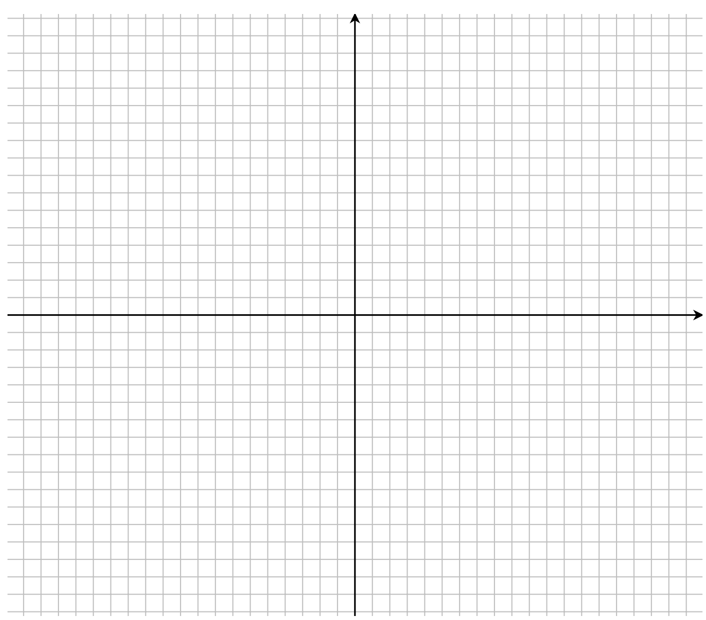
3. Express the given quantity as a single logarithm: $\ln 5 + 5 \ln 3$.

4. Consider the function $f(x) = \ln(x+3)$. Give the domain, range, and x-intercept in the spaces provided, and sketch the graph on the axes provided. Make sure to provide scale on each axis.

   Domain: ______________________

   Range: ______________________

   x-intercept: ______________________



5. If a bacteria population starts with 100 bacteria and doubles every hour, then the number of bacteria after t hours is given by $P(t) = 100 \cdot 2^t$.

   (a) Find the inverse of this function and explain its meaning. (Hint: Solve for t in terms of P.)

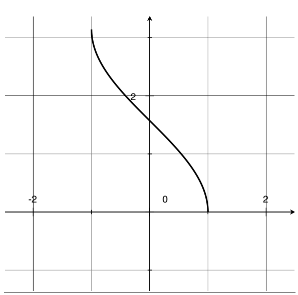
   (b) When will the population reach 400?

6. Without using a calculator, find the exact value of

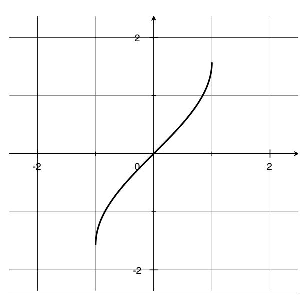
   (a) $\arctan(1)$

   (b) $\cos^{-1}(\sqrt{3}/2)$

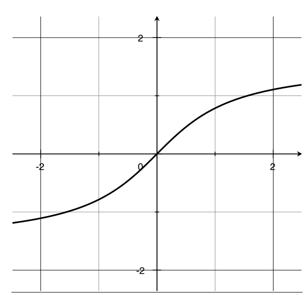
7. Identify the inverse trig. functions below.



(a) Graph of __________



(b) Graph of __________



(c) Graph of __________

2. Composition of Functions

It is relatively straightforward to combine functions by adding them, subtracting them, multiplying them, and dividing them. For example if $f(x) = x^2$ and $g(x) = e^x$,

$$\begin{aligned}
(f+g)(x) &= x^2 + e^x \\
(f-g)(x) &= x^2 - e^x \\
(f\cdot g)(x) &= x^2\, e^x \\
\left(\frac{f}{g}\right)(x) &= \frac{x^2}{e^x} = x^2\, e^{-x}
\end{aligned}$$

To combine functions by **composition**, however, is a bit more subtle. The composition $f \circ g$ of two functions is obtained by applying the two functions one after the other in this order:

$$x \;\mapsto\; g(x) \;\mapsto f(g(x))$$

Using the example functions above:

$$x \;\mapsto\; e^x \;\mapsto (e^x)^2$$

Since $(e^x)^2 = e^{2x}$, we have

$$(f\circ g)(x) = f(g(x)) = (e^x)^2 = e^{2x}$$

Exercises

8. With $f(x) = x^2$ and $g(x) = e^x$, as above, find a formula for $g \circ f$.

9. With $f(x) = \frac{1}{x-1}$ and $g(x) = \cos(x)$, find formulas for $f \circ g$ and $g \circ f$.

10. The function $F(x) = \sin(x^2)$ can be written as the composition $f \circ g$ of two functions $f(x)$ and $g(x)$. Find formulas for f and g. Also find a formula for $g \circ f$.

11. Express the function $G(x) = \sqrt{\frac{x+1}{x-1}}$ as the composition $f \circ g$ of two functions $f(x)$ and $g(x)$. Then find a formula for $g \circ f$.