Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Approximating slope of tangent line by slopes of secant lines
2. Slope function (difference quotient)
3. Approximating instantaneous rates of change using average rates of change

1. The Slope Function

Consider a function $f(x)$ and a number a in its domain. We wish to estimate the slope of the tangent line to the graph of $f(x)$ at $x = a$. We can estimate the slope of the tangent line using secant lines. For a small number h, the slope of the secant line through $P = (a, f(a))$ and $Q = (a + h, f(a + h))$ is:

$$D = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

This is called the *difference quotient*. The value of D depends on h. When we consider D as a function of h, we call it the *slope function* $D(h)$. As h gets closer and closer to zero, the x-value of the point Q gets closer to the x-value of the point P.

Exercises

1. Let $f(x) = \sqrt{x}$ and $a = 4$. We wish to approximate the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

   (a) Use *Mathematica* to compute the D-values, and record your answers, keeping six digits, in the following tables.

h	D
1	
0.1	
0.01	
0.001	

h	D
-1	
-0.1	
-0.01	
-0.001	

   (b) What trends do you notice? What happens to D as h gets close to zero? Does it matter whether h is positive or negative?

(c) According to your computations in (a), what is the slope of the secant line through the points $(4, 2)$ and $(4.1, \sqrt{4.1})$? (No new computations are needed.)

(d) According to your computations in (a), what is the slope of the secant line through the points $(4, 2)$ and $(3.9, \sqrt{3.9})$? (No new computations are needed.)

(e) Is it possible to estimate the slope of the tangent line at $x = 4$? If so, give an estimate. If not, explain why not.

2. Now let $f(x) = |x^2 - 4|$ and $a = 2$.

(a) Use *Mathematica* to compute the D-values, and record your answers, keeping six digits, in the following tables.

h	D
1	
0.1	
0.01	
0.001	

h	D
-1	
-0.1	
-0.01	
-0.001	

(b) What trends do you notice? What happens to D as h gets close to zero? Does it matter whether h is positive or negative?

(c) According to your computations in (a), what is the slope of the secant line through the points $(2, 0)$ and $(1, 3)$? (No new computations are needed.)

(d) Is it possible to estimate the slope of the tangent line at $x = 2$? If so, give an estimate. If not, explain why not.

2. Interpretation as Rates of Change

So far we have been thinking of the difference quotient from a geometric point of view, as the slope of a secant line. In applications, we interpret slopes and difference quotients as rates of change.

For example, on the very first day of class we modeled the dependency of the chirp rate of crickets on the temperature by a linear function. The slope of the function is

$$\frac{\text{change in chirp rate (chirps/min)}}{\text{change in temperature (°F)}} = 3.75 \ \text{(chirps/min)/°F}$$

This is the rate at which the chirp rate changes with respect to the temperature. This means that when the temperature increases by one degree, the chirp rate will increase by 3.75 chirps/min, according to our model. Since our model is linear, the rate of change is constant. (The slope of a straight line is constant.)

In general, the rate of change will vary. In such cases it is necessary to distinguish average rates of change from instantaneous rates of change. One important example is velocity: the rate of change of position with respect to time.

Suppose $s(t)$ represents the position of an object at a given time. Then the average velocity of the object is a difference quotient, which can be written in several different ways:

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{s(t_0 + h) - s(t_0)}{h}$$

The instantaneous velocity at time $t = t_0$ is approximated by the average velocity over a short time interval containing t_0.

In general, if $Q(x)$ represents a quantity, depending on another quantity x, then the average rate of change of Q with respect to x is given by a difference quotient:

$$r_{\text{ave}} = \frac{\Delta Q}{\Delta x} = \frac{Q(x_1) - Q(x_0)}{x_1 - x_0} = \frac{Q(x_0 + h) - Q(x_0)}{h}$$

The instantaneous rate of change when $x = x_0$ of $Q(x)$ with respect to x is approximated by the average rate fo change over a short x-interval containing x_0.

Exercises.

3. A buoy in Lake Superior rises and falls with the waves. Suppose the vertical position of the buoy is given by the formula $s(t) = 3 \sin t$, where s is given in meters and t in seconds.

   (a) Use *Mathematica* to compute the average velocity of the buoy on the given time intervals, and record your answers, keeping six digits, in the following tables.

$[t_0, t_1]$	v_{ave}
$[0, 1]$	
$[0, 0.1]$	
$[0, 0.01]$	

$[t_0, t_1]$	v_{ave}
$[-1, 0]$	
$[-0.1, 0]$	
$[-0.01, 0]$	

   (b) Use your work to estimate the instantaneous velocity of the buoy at $t = 0$. Make sure to include units.

4. Let $T(t)$ be the temperature (in °F) in Phoenix t hours after midnight on September 10, 2008. The table shows values of this function recorded every two hours.

t	0	2	4	6	8	10	12	14
T	82	75	74	75	84	90	93	94

(a) How much did the temperature change from 6 am to 8 am? What is the average rate of change of the temperature over this time interval? Make sure to include units in your answer.

(b) How much did the temperature change from 8 am to 10 am? What is the average rate of change of the temperature over this time interval? Make sure to include units in your answer.

(c) Use your results from (a) and (b) to give an approximation for the rate at which the temperature was changing at 8 am.