Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Careful definition of continuity using limits
2. Theorem about continuity of some familiar functions
3. Intermediate Value Theorem (IVT)

1. Definition of Continuity

Recall our intuitive notion of continuity: a function $f(x)$ is continuous at a point $x = a$ if $f(x)$ is close to $f(a)$ when x is close to a. For example, we are implicitly using the continuity of $f(x) = \sqrt{x}$ at $x = 4$, when we say that $\sqrt{4.1}$ should be about $\sqrt{4}$ (i.e. 2), since 4.1 is close to 4.

Now that we discussed the notion of a limit, we can use limits to make this intuitive notion more precise:

A function f is continuous at a number a if: $\lim_{x \to a} f(x) = f(a)$

Notice that this definition implicitly requires three things:

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x)$ equals $f(a)$

Note that (2) implicitly requires three things as well: that the limit from the left exists, the limit from the right exists, and both of these limits agree.

So, to show that a function is continuous at a given point, we must show that (1), (2), and (3) are all true.

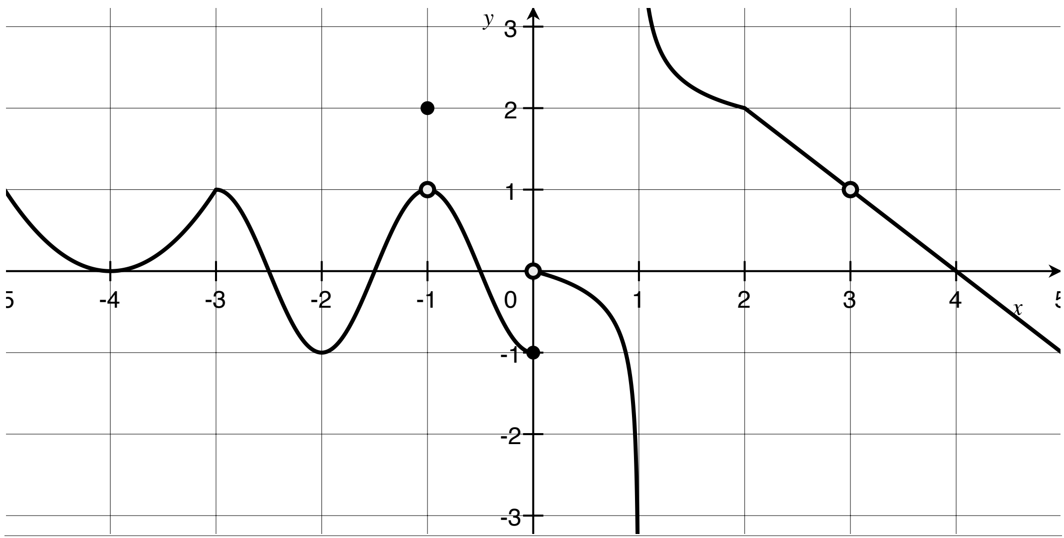
To show that a function is *discontinuous* at a given point, we must demonstrate that one or more of the conditions (1), (2) or (3) is violated. (See Examples 1 and 2 on page 119.)

See the first full paragraph on page 120 for a description of three kinds of discontinuities: removable discontinuities, infinite discontinuities, and jump discontinuities.

We say that a function is continuous on an interval (a, b) if it is continuous for all numbers in the interval. In particular, a function is continuous on $(-\infty, \infty)$ if it is continuous for all real numbers. In this case, we say that it is continuous (period).

Exercises

1. Consider the function $f(x)$ in the graph below, and fill in the table to describe the behavior of the function at the specified points. Write "DNE" if the specified value does not exist. In the column where you are asked "why", state which condition(s), (1), (2), or (3) from above, is violated. In the last column, classify the discontinuity as a removable discontinuity, an infinite discontinuity, or a jump discontinuity.



x_o	$\lim_{x\to x_o^-} f(x)$	$\lim_{x\to x_o^+} f(x)$	$\lim_{x\to x_o} f(x)$	$f(x_o)$	Continuous? Y/N	If no, why?	What kind?
-3							
-2							
-1							
0							
1							
2							
3							
4							

2. Use the definition of continuity to show that function $f(x) = x^2 + 2$ is continuous at $x = -2$. (You need to check (1), (2), and (3).)

3. Use the definition of continuity to show that function below is continuous at $x = 1$. (Again, check (1), (2), and (3). This time you will need to use left and right limits to find the limit as $x \to 1$.)

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

2. Continuity of Some Familiar Functions

The familiar functions that we have discussed are important both because they model real-life phenomena that we care about and because they are relatively well-behaved. In particular, they don't have jump discontinuities. Some of them (for example, many rational functions) do have removable discontinuities and/or infinite discontinuities, as in the exercise above. However, these discontinuities only occur at numbers that are not in the domain of the original function. In other words:

> **Theorem:** All polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions, and logarithmic functions are continuous where defined.

Look over pages 121-123 for a justification of this theorem.

Exercise:

4. Show that the function below is continuous on $(-\infty, \infty)$.

$$f(x) \quad = \quad \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

(Hint: You need to show that this function is continuous for every real number a. To do this, break it down into three cases: (i) $a < 1$, (ii) $a > 1$, and (iii) $a = 1$. For the first two cases, use the theorem above. For the third case, just reference Exercise 3!)