Math 113, Prep 2.5B: Continuity

Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Proving discontinuity
2. Intermediate Value Theorem (IVT)

1. Proving Discontinuity

Recall that a function f is continuous at a number a if: $\lim_{x \to a} f(x) = f(a)$, i.e.

(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x)$ equals $f(a)$

To show that a function is *discontinuous* at a given point, we must demonstrate that one or more of the conditions (1), (2) or (3) is violated. (See Examples 1 and 2 on page 119.)

Exercises

1. Use the definition of continuity to explain why the function below is discontinuous at $x = -2$.

$$f(x) = \begin{cases} \dfrac{(x+2)}{(x+2)(x-1)} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

Hint: To show that the function is *not* continuous at $x = -2$, you need to show that one of the conditions (1), (2), or (3) is violated. This requires up to three steps: first find $f(-2)$, then compute the limit as $x \to -2$, then compare the two and draw your conclusion.

2. We can change the function from the previous exercise to make it continuous, by changing the 1 to another number. What number c would make the function continuous at $x = -2$?

$$f(x) \;=\; \begin{cases} \dfrac{(x+2)}{(x+2)(x-1)} & \text{if } x \neq -2 \\ c & \text{if } x = -2 \end{cases}$$

   This is called "removing the discontinuity."

3. Consider the function $f(x) \;=\; \dfrac{1-x^2}{2-x-x^2}$.

   (a) Use the definition of continuity to explain why $f(x)$ is discontinuous at $x = -2$. (This may require three steps, but think about whether you really need all three steps in this case.)

   (b) Use the definition of continuity to explain why $f(x)$ is discontinuous at $x = 1$. (Again, think about whether you really need all three steps in this case.)

   (c) What kind of discontinuity (jump, removable, or infinite) is at $x = -2$? How can you tell? (Hint: factor the numerator and denominator.)

   (d) What kind of discontinuity (jump, removable, or infinite) is at $x = 1$? How can you tell?

2. The Intermediate Value Theorem

Common sense tells us that if some quantity, say temperature, changes in a continuous way with an initial value of, say 52°, and a final value of, say 81°, then for any value between those the inital and final values, say a temperature of 68°, there must have been a moment when the quantity was exactly equal to that intermediate value, 68°.

This idea is formalized in the **Intermediate Value Theorem (IVT)**:

> Suppose f is continuous on a closed interval $[a, b]$. Suppose the y-values $f(a)$ and $f(b)$ are not equal. Pick any number N between $f(a)$ and $f(b)$. Then there is a number c in (a, b) such that $f(c) = N$.

Notice that the IVT does *not* tell you *anything* about the number c, except that it is between a and b! So you should not expect to find c, unless you are *specifically* asked to find it.

The IVT is often used to show that a given equation has a root in between two specified values. See Example 10, page 126.

Exercise:

4. Use the IVT to show that there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.
   (a) Let $f(x) = x^4 + x - 3$. We wish to use the IVT to show that there is a number c in $(1, 2)$ such that $f(c) = 0$. In this example, what are the values for a, b, and N in the statement of the IVT?

   (b) In order to use the IVT, we need to demonstrate that it is relevant; namely, we must verify the hypotheses of the theorem:
      i. Explain why $f(x)$ is continuous on the interval $[1, 2]$.

      ii. Show that $f(a) \neq f(b)$.

      iii. Check that N is between $f(a)$ and $f(b)$.

   (c) Now that we have verified the hypotheses of the theorem, we can use the theorem to draw a conclusion. What does the IVT allow us to conclude?