Math 113, Prep 3.5C, 3.6C: Implicit Differentiation and Logarithmic Differentiation

Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Implicit differentiation
2. Logarithmic differentiation

1. Implicit Differentiation

So far we have looked at finding $\frac{dy}{dx}$ when y is defined explicitly by a function of x, i.e. $y = f(x)$. Now we will look at finding $\frac{dy}{dx}$ when the relationship between x and y might not be so simple. For example, we might have an equation with x's and y's on both sides, and it might not be possible to get the y on a side by itself. This means that y is defined *implicitly*. The method of finding $\frac{dy}{dx}$ in such a case is called *implicit differentiation*. See Examples 1-3 in 3.5.

Implicit Differentiation:

1. Take $\frac{d}{dx}$ of both sides.
2. Take derivatives remembering that y is a function of x. Use the product rule, quotient rule, chain rule where appropriate. For example:

   Product Rule: $\frac{d}{dx}\big((\sin x)(y)\big) = (\cos x)(y) + (\sin x)(\frac{dy}{dx})$

   Quotient Rule: $\frac{d}{dx}(\frac{x^2}{y}) = \frac{2xy - x^2(\frac{dy}{dx})}{y^2}$

   Chain Rule: $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$

3. Get $\frac{dy}{dx}$ on a side by itself.

Exercises

1. Use implicit differentiation to find $\frac{dy}{dx}$ when $4x^2 + 9y^2 = 36$.

2. Find y' when $y^5 + x^2 y^3 = 1 + ye^{x^2}$

3. Find the tangent line to the curve $\sqrt{x} + \sqrt{y} = 4$ at the point $(4, 4)$.

2. Logarithmic Differentiation

Logarithmic differentiation is a technique we apply to particularly nasty functions when we want to differentiate them.

Remember that the log takes a *product* and gives us a *sum*, and when it comes to taking derivatives, we like sums better than products! Similarly, a log takes a *quotient* and gives us a *difference*. Again, when it comes to taking derivatives, we'd much prefer a difference to a quotient. Finally, the log takes something of the form a^b and gives us the product $b \log a$. This can be an improvement when it comes to differentiation.

Logarithmic Differentiation: Suppose you have $y = f(x)$ and $f(x)$ is a nasty combination of products, quotients, etc. The trick is to:

- Apply the natural log to both sides: $\ln y = \ln(f(x))$, and use the laws of logs to simplify the right hand side as much as possible.
- Take the derivative (with respect to x) of both sides. You have to use the chain rule on the left side:

$$\frac{y'}{y} = (\text{RHS})'$$

- Solve for y' by multiplying both sides by the original function:

$$y' = f(x) \cdot (\text{RHS})'$$

See 3.6, Examples 7 and 8.

Exercises

4. Consider the function $y = \sqrt{x}\, e^{x^2}(x^2+1)^{10}$.

   (a) Apply the natural log to both sides and use the log laws to simplify the right hand side, namely $\ln\left(\sqrt{x}\, e^{x^2}(x^2+1)^{10}\right)$, as much as possible.

   (b) Apply $\frac{d}{dx}$ to both sides, use implicit differentiation, and solve for y'.

5. Use logarithmic differentiation to find the derivative y' of $y = x^{1/x}$.

6. In this exercise, we look at four cases for exponents and bases.

   (c) Consider the function $f(x) = x^{9/2}$. What kind of function is this? What is the name of the differentiation rule that is used to find the derivative? What is $f'(x)$?

   (d) Consider the function $g(x) = (5/4)^x$. What kind of function is this? What is $g'(x)$?

   (e) Consider the function $h(x) = e^{\pi}$. What kind of function is this? What is $h'(x)$?

   (f) Consider the function $j(x) = x^x$. Is this function in our catalog of familiar functions? What technique can be used to differentiate $j(x)$? What is $j'(x)$?