Math 113, Prep 5.5: The Technique of Substitution

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. Undoing the Chain Rule
2. More complicated problems
3. Substitution and definite integrals

1. Undoing the Chain Rule

So far the only strategy we have for finding antiderivatives is to recognize them as derivatives of familiar functions. For example, an antiderivative of $\cos x$ is $\sin x$, because $\cos x$ is the derivative of $\sin x$, and $\frac{1}{x^2+1}$ is an antiderivative for $\arctan x$, since the derivative of $\arctan x$ is $\frac{1}{x^2+1}$.

Sometimes it is necessary to use algebra or trigonometry to rewrite a function before we can recognize it as a derivative of a familiar function. For example, we rewrite $\sqrt{z}(z+1/z)$ as $z^{3/2}+z^{-1/2}$ in order to recognize it as the derivative of $\frac{2}{5}z^{5/2}+2z^{1/2}$.

Can you recognize $2e^{2x}$ as the derivative of a familiar function? It is the derivative of e^{2x}. The constant factor of 2 comes from the chain rule.

Exercises

1. Find an antiderivative for the following functions, by guess and check. Hint: "undo the chain rule."

   (a) $f(x) = \pi \cos(\pi x)$

   (b) $g(x) = 2x\, e^{x^2}$

2. Evaluate the indefinite integrals:

   (a) $\int 3\sin(3x)\, dx$

   (b) $\int e^{5x}\, dx$

Sometimes it helps to give the "inside function" a name. Usually we use $u(x)$ for the inside function. The chain rule says that

$$\frac{d}{dx} F(u(x)) \quad = \quad F'(u(x)) \cdot u'(x)$$

Thus, if F is an antiderivative for f (i.e. $F' = f$),

$$\int f(u)\, u'(x)\, dx \;=\; \int f(u)\, du \;=\; F(u(x)) \;+\; C$$

since $u'(x)\, dx = du$, as we remember from the section on differentials. See Examples 1-3, pages 408-409.

Exercise

3. Evaluate the integrals using substitution. State what u and du are, and clearly transform the original integral into a new integral in terms of u. Your final answer should be in terms of x.

   (a) $\int (1-2x)^9\, dx$

   (b) $\int t\sqrt{1-t^2}\, dt$

   (c) $\displaystyle\int \frac{z^2}{z^3+1}\, dz$

   (d) $\int \cos^4(\theta)\, \sin\theta\, d\theta$

2. More complicated problems

We can sometimes use substitution even if the integrand is not a constant multiple of something of the form $f(u(x))\,u'(x)$. In particular, as long as the integrand can be rewritten as $u'(x)$ times something entirely in terms of u, substitution is worth trying. See Example 5.

Exercises

4. Evaluate the integral $\int x(x+5)^8\,dx$. (Hint: Let $u = x+5$ and notice that $x = u-5$.)

5. Evaluate the integral $\int x^2\sqrt{2+x}\,dx$.

6. Evaluate the integral $\displaystyle\int \frac{x}{x+1}\,dx$.

3. Substitution and the definite integral

Since substitution is a technique for finding antiderivatives, it is also useful for definite integrals. The trick is to be careful not to plug in x-values for u.

Exercises

7. Use an antiderivative for $f(x) = \pi\cos(\pi x)$ to evaluate $\int_0^1 \pi\cos(\pi x)\,dx$. (See 1a.)

8. Use an antiderivative for $g(x) = (1 - 2x)^9$ to evaluate $\int_0^1 (1 - 2x)^9 \, dx$. (See 3a.)

Evaluating definite integrals in this way can be cumbersome, because you need to stop and find an antiderivative before plugging in the upper and lower limits of integration. As an alternative, you can transform the limits of integration along with the whole integral. This is called "The Substitution Rule for Definite Integrals." Read page 411 to see how this works.

Exercise

9. Evaluate the definite integrals using substitution to transform the definite integral to a new definite integral with new limits of integration, as in Examples 7, 8, and 9.

   (a) $\int_0^{2/3} (3t - 1)^{10} \, dt$

   (b) $\displaystyle\int_0^3 \frac{dx}{5x + 1}$

   (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1 + 2x}} \, dx$