(b) Find the critical points of f.

(c) Find any inflection points of f.

(d) Evaluate f at its critical points and at the endpoints of the given interval. Identify local and global maxima and minima of f in the interval.

(e) Graph f.

18. $f(x) = x^3 - 3x^2 \quad (-1 \le x \le 3)$

19. $f(x) = 2x^3 - 9x^2 + 12x + 1 \, (-0.5 \le x \le 3)$

20. $f(x) = x^3 - 3x^2 - 9x + 15 \quad (-5 \le x \le 4)$

21. $f(x) = x + \sin x \quad (0 \le x \le 2\pi)$

22. $f(x) = e^{-x} \sin x \quad (0 \le x \le 2\pi)$

In Problems 23–28, find the exact global maximum and minimum values of the function. The domain is all real numbers unless otherwise specified.

23. $g(x) = 4x - x^2 - 5$

24. $f(x) = x + 1/x$ for $x > 0$

25. $g(t) = te^{-t}$ for $t > 0$

26. $f(x) = x - \ln x$ for $x > 0$

27. $f(t) = \dfrac{t}{1+t^2}$

28. $f(t) = (\sin^2 t + 2) \cos t$

29. Find the value(s) of x that give critical points of $y = ax^2 + bx + c$, where a, b, c are constants. Under what conditions on a, b, c is the critical value a maximum? A minimum?

30. What value of w minimizes S if $S - 5pw = 3qw^2 - 6pq$ and p and q are positive constants?

31. Figure 4.39 gives the derivative of $g(x)$ on $-2 \le x \le 2$.

(a) Write a few sentences describing the behavior of $g(x)$ on this interval.

(b) Does the graph of $g(x)$ have any inflection points? If so, give the approximate x-coordinates of their locations. Explain your reasoning.

(c) What are the global maxima and minima of g on $[-2, 2]$?

(d) If $g(-2) = 5$, what do you know about $g(0)$ and $g(2)$? Explain.

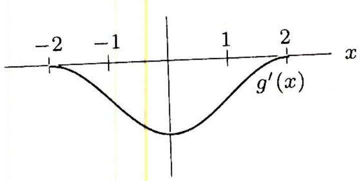


Figure 4.39

32. The energy expended by a bird per day, E, depends on the time spent foraging for food per day, F hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense.[4] Find the foraging time that minimizes energy expenditure if

$$E = 0.25F + \frac{1.7}{F^2}.$$

33. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?

34. A closed box has a fixed surface area A and a square base with side x.

(a) Find a formula for its volume, V, as a function of x.

(b) Sketch a graph of V against x.

(c) Find the maximum value of V.

HH
Bio

35. On the west coast of Canada, crows eat whelks (a shellfish). To open the whelks, the crows drop them from the air onto a rock. If the shell does not smash the first time, the whelk is dropped again.[5] The average number of drops, n, needed when the whelk is dropped from a height of x meters is approximated by

$$n(x) = 1 + \frac{27}{x^2}.$$

(a) Give the total vertical distance the crow travels upward to open a whelk as a function of drop height, x.

(b) Crows are observed to drop whelks from the height that minimizes the total vertical upward distance traveled per whelk. What is this height?

36. During a flu outbreak in a school of 763 children, the number of infected children, I, was expressed in terms of the number of susceptible (but still healthy) children, S, by the expression[6]

$$I = 192 \ln\left(\frac{S}{762}\right) - S + 763.$$

What is the maximum possible number of infected children?

HH
Ag

37. An apple tree produces, on average, 400 kg of fruit each season. However, if more than 200 trees are planted per km^2, crowding reduces the yield by 1 kg for each tree over 200.

(a) Express the total yield from one square kilometer as a function of the number of trees on it. Graph this function.

(b) How many trees should a farmer plant on each square kilometer to maximize yield?

[4] Adapted from Graham Pyke, reported by J. R. Krebs and N. B. Davis in *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

[5] Adapted from Reto Zach, reported by J. R. Krebs and N. B. Davis in *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

[6] Data from Communicable Disease Surveillance Centre (UK), reported in "Influenza in a Boarding School", *British Medical Journal*, March 4, 1978.

38. The number of offspring in a population may not be a linear function of the number of adults. The Ricker curve, used to model fish populations, claims that $y = axe^{-bx}$, where x is the number of adults, y is the number of offspring, and a and b are positive constants.

(a) Find and classify all critical points of the Ricker curve.

(b) Is there a global maximum? What does this imply about populations?

39. The oxygen supply, S, in the blood depends on the hematocrit, H, the percentage of red blood cells in the blood:

$$S = aHe^{-bH} \quad \text{for positive constants } a, b.$$

(a) What value of H maximizes the oxygen supply? What is the maximum oxygen supply?

(b) How does increasing the value of the constants a and b change the maximum value of S?

40. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given, in mg, by

$$q(t) = 20(e^{-t} - e^{-2t}).$$

(a) How much of the drug is in the bloodstream at time $t = 0$?

(b) When is the maximum quantity of drug in the bloodstream? What is that maximum?

(c) In the long run, what happens to the quantity?

41. When birds lay eggs, they do so in clutches of several at a time. When the eggs hatch, each clutch gives rise to a brood of baby birds. We want to determine the clutch size which maximizes the number of birds surviving to adulthood per brood. If the clutch is small, there are few baby birds in the brood; if the clutch is large, there are so many baby birds to feed that most die of starvation. The number of surviving birds per brood as a function of clutch size is shown by the benefit curve in Figure 4.40.[7]

(a) Estimate the clutch size which maximizes the number of survivors per brood.

(b) Suppose also that there is a biological cost to having a larger clutch: the female survival rate is reduced by large clutches. This cost is represented by the dotted line in Figure 4.40. If we take cost into account by assuming that the optimal clutch size in fact maximizes the vertical distance between the curves, what is the new optimal clutch size?

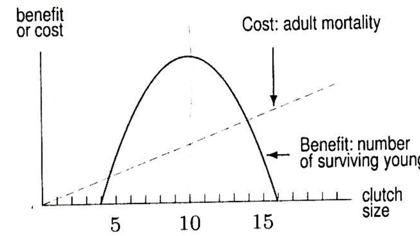


Figure 4.40

42. Let $f(v)$ be the amount of energy consumed by a flying bird, measured in joules per second (a joule is a unit of energy), as a function of its speed v (in meters/sec). Let $a(v)$ be the amount of energy consumed by the same bird, measured in joules per meter.

(a) Suggest a reason (in terms of the way birds fly) for the shape of the graph of $f(v)$ in Figure 4.41.

(b) What is the relationship between $f(v)$ and $a(v)$?

(c) Where is $a(v)$ a minimum?

(d) Should the bird try to minimize $f(v)$ or $a(v)$ when it is flying? Why?

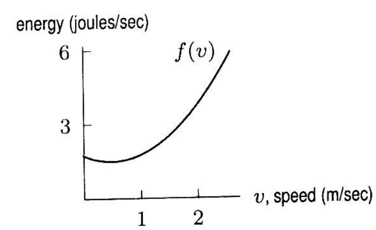


Figure 4.41

HH

43. As an epidemic spreads through a population, the number of infected people, I, is expressed as a function of the number of susceptible people, S, by

Bio

$$I = k\ln\left(\frac{S}{S_0}\right) - S + S_0 + I_0, \quad \text{for } k, S_0, I_0 > 0.$$

(a) Find the maximum number of infected people.

(b) The constant k is a characteristic of the particular disease; the constants S_0 and I_0 are the values of S and I when the disease starts. Which of the following affects the maximum possible value of I? Explain.
- The particular disease, but not how it starts.
- How the disease starts, but not the particular disease.
- Both the particular disease and how it starts.

44. The hypotenuse of a right triangle has one end at the origin and one end on the curve $y = x^2e^{-3x}$, with $x \geq 0$. One of the other two sides is on the x-axis, the other side is parallel to the y-axis. Find the maximum area of such a triangle. At what x-value does it occur?

[7] Data from C. M. Perrins and D. Lack, reported by J. R. Krebs and N. B. Davies in *An Introduction to Behavioural Ecology* (Oxford: Blackwell, 1987).

25. **(a)** Production of an item has fixed costs of \$10,000 and variable costs of \$2 per item. Express the cost, C, of producing q items.
(b) The relationship between price, p, and quantity, q, demanded is linear. Market research shows that 10,100 items are sold when the price is \$5 and 12,872 items are sold when the price is \$4.50. Express q as a function of price p.
(c) Express the profit earned as a function of q.
(d) How many items should the company produce to maximize profit? (Give your answer to the nearest integer.) What is the profit at that production level?

HH

min. cost.

26. A landscape architect plans to enclose a 3000 square-foot rectangular region in a botanical garden. She will use shrubs costing \$45 per foot along three sides and fencing costing \$20 per foot along the fourth side. Find the minimum total cost.

HH max. revenue

27. You run a small furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be \$90 per chair up to 300 chairs, and above 300, the price will be reduced by \$0.25 per chair (on the whole order) for every additional chair over 300 ordered. What are the largest and smallest revenues your company can make under this deal?

HH min cost. *

28. A warehouse selling cement has to decide how often and in what quantities to reorder. It is cheaper, on average, to place large orders, because this reduces the ordering cost per unit. On the other hand, larger orders mean higher storage costs. The warehouse always reorders cement in the same quantity, q. The total weekly cost, C, of ordering and storage is given by

$$C = \frac{a}{q} + bq, \quad \text{where } a, b \text{ are positive constants.}$$

(a) Which of the terms, a/q and bq, represents the ordering cost and which represents the storage cost?
(b) What value of q gives the minimum total cost?

29. A business sells an item at a constant rate of r units per month. It reorders in batches of q units, at a cost of $a + bq$ dollars per order. Storage costs are k dollars per item per month, and, on average, $q/2$ items are in storage, waiting to be sold. [Assume r, a, b, k are positive constants.]

(a) How often does the business reorder?
(b) What is the average monthly cost of reordering?
(c) What is the total monthly cost, C of ordering and storage?
(d) Obtain Wilson's lot size formula, the optimal batch size which minimizes cost.

HH max * revenue

30. **(a)** A cruise line offers a trip for \$2000 per passenger. If at least 100 passengers sign up, the price is reduced for *all* the passengers by \$10 for every additional passenger (beyond 100) who goes on the trip. The boat can accommodate 250 passengers. What number of passengers maximizes the cruise line's total revenue? What price does each passenger pay then?
(b) The cost to the cruise line for n passengers is $80{,}000 + 400n$. What is the maximum profit that the cruise line can make on one trip? How many passengers must sign up for the maximum to be reached and what price will each pay?

31. A company manufactures only one product. The quantity, q, of this product produced per month depends on the amount of capital, K, invested (i.e., the number of machines the company owns, the size of its building, and so on) and the amount of labor, L, available each month. We assume that q can be expressed as a *Cobb-Douglas production function*:

$$q = cK^{\alpha}L^{\beta}$$

where c, α, β are positive constants, with $0 < \alpha < 1$ and $0 < \beta < 1$. In this problem we will see how the Russian government could use a Cobb-Douglas function to estimate how many people a newly privatized industry might employ. A company in such an industry has only a small amount of capital available to it and needs to use all of it, so K is fixed. Suppose L is measured in man-hours per month, and that each man-hour costs the company w rubles (a ruble is the unit of Russian currency). Suppose the company has no other costs besides labor, and that each unit of the good can be sold for a fixed price of p rubles. How many man-hours of labor per month should the company use in order to maximize its profit?

32. A company can produce and sell $f(L)$ tons of a product per month using L hours of labor per month. The wage of the workers is w dollars per hour, and the finished product sells for p dollars per ton.

(a) The function $f(L)$ is the company's production function. Give the units of $f(L)$. What is the practical significance of $f(1000) = 400$?
(b) The derivative $f'(L)$ is the company's marginal product of labor. Give the units of $f'(L)$. What is the practical significance of $f'(1000) = 2$?
(c) The real wage of the workers is the quantity of product that can be bought with one hour's wages. Show that the real wage is w/p tons per hour.
(d) Show that the monthly profit of the company is

$$\pi(L) = pf(L) - wL.$$

(e) Show that when operating at maximum profit, the company's marginal product of labor equals the real wage:

$$f'(L) = \frac{w}{p}.$$

Use optimization techniques to answer the questions in Exercises 25–30.

25. Find two real numbers x and y whose sum is 36 and whose product is as large as possible.

26. Find two real numbers x and y whose sum is 36 and whose product is as small as possible.

27. Find real numbers a and b whose sum is 100 and for which the sum of the squares of a and b is as small as possible.

28. Find the area of the largest rectangle that fits inside a circle of radius 4.

29. Find the area of the largest rectangle that fits inside a circle of radius 10.

30. Find the volume of the largest cylinder that fits inside a sphere of radius 10.

In Exercises 31–34, find the point on the graph of the function f that is closest to the point (a, b) by minimizing the square of the distance from the graph to the point.

31. $f(x) = 3x + 1$ and the point $(-2, 1)$

32. $f(x) = x^2$ and the point $(0, 3)$

33. $f(x) = x^2 - 2x + 1$ and the point $(1, 2)$

34. $f(x) = \sqrt{x^2 + 1}$ and the point $(2, 0)$

Applications

A farmer wants to build four fenced enclosures on his farmland for his free-range ostriches. To keep costs down, he is always interested in enclosing as much area as possible with a given amount of fence. For the fencing projects in Exercises 35–38, determine how to set up each ostrich pen so that the maximum possible area is enclosed, and find this maximum area.

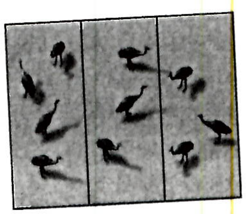
K
n
st

35. A rectangular ostrich pen built with 350 feet of fencing material.

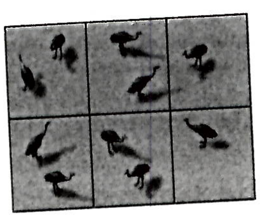
36. A rectangular ostrich pen built along the side of a river (so that only three sides of fence are needed), with 540 feet of fencing material.

37. A rectangular ostrich pen built with 1000 feet of fencing material, divided into three equal sections by two interior fences that run parallel to the exterior side fences, as shown next at the left.

Ostrich pen with three sections



Ostrich pen with six sections



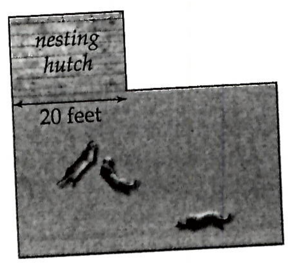
38. A rectangular ostrich pen that is divided into six equal sections by two interior fences that run parallel to the east and west fences, and another interior fence running parallel to the north and south fences, as shown previously at the right. The farmer has allotted 2400 feet of fencing material for this important project.

You are in charge of constructing a zoo habitat for prairie dogs, with the requirement that the habitat must enclose 2500 square feet of area and use as little border fencing as possible. For each of the habitat designs described in Exercises 39–42, find the amount of border fencing that the project will require.

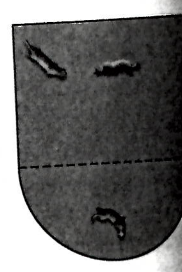
TK
min
cost

39. A rectangular habitat with a 20-foot-wide nesting hutch along one side (so that fencing is not needed along those 20 feet), as shown next at the left.

Rectangular habitat with hutch



Arena-style habitat



TK
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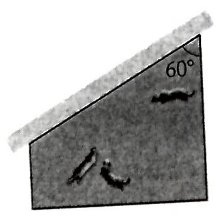
40. An arena-style habitat whose front area is a semicircle and whose back area is rectangular, as shown previously at the right.

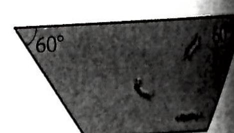
TK
*

41. A trapezoid-shaped habitat whose angled side is an enclosed walkway for zoo patrons (so that no fencing is needed along the walkway), where the walkway makes an angle of 60° with the right fence, as shown next at the left.

Habitat along walkway



Habitat with mural



TK
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42. An arena-style trapezoid-shaped habitat whose long back side is a wall with a landscape mural (so no fencing is needed along the back wall), where the back wall makes an angle of 60° with the slanted side fences, as shown previously at the right.

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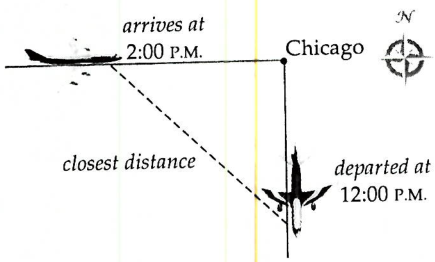
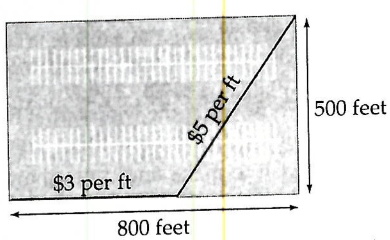


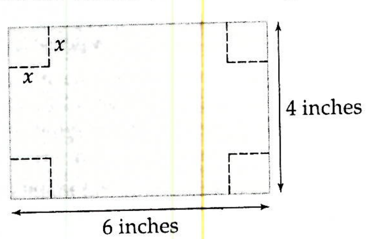
57. The cost of the material for the top and bottom of a cylindrical can is 5 cents per square inch. The material for the rest of the can costs only 2 cents per square inch. If the can must hold 400 cubic inches of liquid, what is the cheapest way to make the can? What is the most expensive way?

58. Consider the can-making situation in the previous exercise, but suppose that the cans are made with open tops. If each can must hold 400 cubic inches of liquid, what is the cheapest way to make the cans? What is the most expensive way?

* **59.** A steam pipe must be buried underground to reach from one corner of a rectangular parking lot to the diagonally opposite corner. The dimensions of the parking lot are 500 feet by 800 feet. It costs 5 dollars per foot to lay steam pipe under the pavement but only 3 dollars per foot to lay the pipe along one of the long edges of the parking lot. Because of nearby sidewalks, the pipe cannot be laid along the 500-foot sides of the parking lot. How should the steam pipe be buried so as to cost as little as possible?



60. Suppose you want to make an open-topped box out of a 4×6 index card by cutting a square out of each corner and then folding up the edges, as shown in the figure. How large a square should you cut out of each corner in order to maximize the volume of the resulting box?



61. Your family makes and sells velvet Elvis paintings. After many years of research you have found a function that predicts how many paintings you will sell in a year, based on the price that you charge per painting. You always charge between \$5.00 and \$55.00 per painting. Specifically, if you charge c dollars per painting, then you can sell $N(c) = 0.6c^2 - 54c + 1230$ paintings in a year.

(a) What price should you charge to sell the *greatest* number of velvet Elvis paintings, and how many could you sell at that price? For what price would you sell the *least* number of paintings, and how many would you sell?

(b) Write down a function that predicts the revenue $R(c)$, in dollars, that you will earn in a year if you charge c dollars per painting. (*Hint: Try some examples first; for example, what would your yearly revenue be if you charged* \$10.00 *per painting? What about* \$50.00? *Then write down a function that works for all values of* c.)

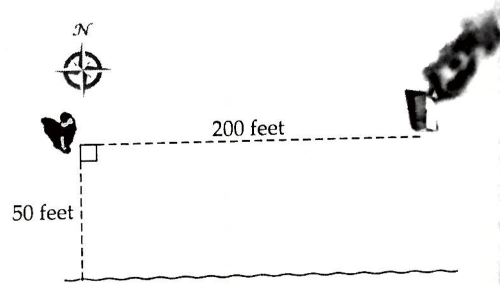
(c) What price should you charge to earn the *most* money, and how much money would you earn? What price per painting would cause you to make the *least* amount of money in a year, and how much money would you make in that case?

(d) Explain why you do not make the most money at the same price per painting for which you sell the most paintings.

TK

Survival *

62. While you are on a camping trip, your tent accidentally catches fire. At the time, you and the tent are both 50 feet from a stream and you are 200 feet away from the tent, as shown in the diagram. You have a bucket with you, and need to run to the stream, fill the bucket, and run to the tent as fast as possible. You can run only half as fast while carrying the full bucket as you can empty handed, and thus any distance travelled with the full bucket is effectively twice as long. Complete parts (a)–(f) to determine how you can put out the fire as quickly as possible.



(a) Let x represent the distance from the point on the stream directly "below" you to the point on the stream that you run to. Sketch the path that you would follow to run from your starting position, to the point x along the stream, to the tent.

(b) Let $D(x)$ represent the effective distance (counting twice any distance travelled while carrying a full bucket) you have to run in order to collect water and get to the tent. Write a formula for $D(x)$.

(c) Determine the interval I of x-values on which $D(x)$ should be minimized. Explain in practical terms what happens at the endpoints of this interval, and calculate the value of $D(x)$ at these endpoints.

(d) Find $D'(x)$, and simplify as much as possible. Are there any points (in the interval I) at which $D'(x)$ is undefined?

↳ continued...

TK #62* (con't)

(e) It is difficult to find the zeroes of $D'(x)$ by hand. Use a graphing utility to approximate any zeroes of $D'(x)$ in the interval I, and test these zeroes by evaluating D' at each one.

(f) Use the preceding information to determine the minimum value of $D(x)$, and then use this value to answer the original word problem.

Proofs

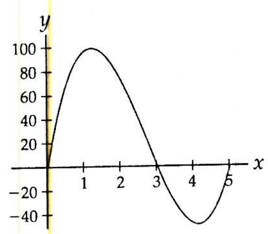
63. Prove that the rectangle with the largest possible area given a fixed perimeter P is always a square.

64. Prove that the most efficient way to build a rectangular fenced area along a river—so that only three sides of fencing are needed—is to make the side parallel to the river twice as long as the other sides. You may assume that you have a fixed amount of fencing material.

Thinking Forward

Consider the graph of the function f shown next. Define $A(x)$ to be the area of the region between the graph of f and the x-axis from 0 to x. We will count areas of regions above the x-axis positively and areas of regions below the x-axis negatively.

$A(x)$ is area under this graph on $[0, x]$



- Use the graph to approximate the values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, and $A(5)$.
- From the graph of f, estimate all local maxima and minima of $A(x)$.
- From the graph of f, estimate all global maxima and minima of $A(x)$, if any.
- It turns out that the function f whose graph is shown is given by the formula $f(x) = 12x^3 - 96x^2 + 180x$ and that the area function A is given by the formula $A(x) = 3x^4 - 32x^3 + 90x^2$. What surprising relationship do f and A have?
- Show that your answers for the local and global extrema of $A(x)$ are reasonable by using optimization techniques on the area function $A(x) = 3x^4 - 32x^3 + 90x^2$.

3.5 RELATED RATES

- Using implicit differentiation to obtain relationships between rates
- Formulas for volume, surface area, and relationships between side lengths of triangles
- Techniques for solving related-rates problems

Related Quantities Have Related Rates

If two quantities that change over time are related to each other, then their rates of change over time will also be related to each other. For example, consider an expanding circle. Clearly the radius $r = r(t)$ of the circle and the area $A = A(t)$ of the circle are related: If you know one of these quantities at some time t, then you also know the other, via the formula $A = \pi r^2$. As the circle expands over time, the rate $\frac{dr}{dt}$ at which its radius increases is related to the rate $\frac{dA}{dt}$ at which its area increases. We can find an equation that relates these rates by applying implicit differentiation to the formula that relates the quantities r and A:

$$A = \pi r^2 \qquad \leftarrow \text{relationship between } A \text{ and } r$$

$$\frac{d}{dt}(A(t)) = \frac{d}{dt}(\pi (r(t))^2) \qquad \leftarrow \text{differentiate both sides with respect to } t$$

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) \qquad \leftarrow \text{chain rule}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}. \qquad \leftarrow \text{relationship between } \frac{dA}{dt} \text{ and } \frac{dr}{dt}$$