Math 113, Prep: Review for Exam 2

Name: amy Section: ______

Typos ~~#1, #6~~, #7, #11b & double check the rest.

Names of collaborators: ______

Main Points:

1. Intervals of increase/decrease, local max/min points, intervals of concavity, inflection points
2. Absolute max/min points, optimization
3. Rolle's theorem, MVT
4. L'Hospital's rule for evaluating limits
5. Estimating accumulated net change over an interval; approximating area under curve with rectangles
6. Exact accumulated net change over an interval; definite integral
7. Area-so-far functions, antiderivatives, FTC, evaluating definite integrals using antiderivatives

Practice Problems:

1. Fill in the blank.

The absolute maximum value of a continuous function on a closed function will always be obtained either at an endpoint or at a local max. Local maximum values are always attained at critical numbers.

If f is a differentiable function such that $f(2) = f(5)$, then Rolle's Theorem implies that, for some number c between 2 and 5, $f'(c) = 0$.

When $C''(s)$ is negative, $C'(s)$ is decreasing, and $C(s)$ is concave down.

True or False: If $\lim_{x\to-1} f(x) = \infty$ and $\lim_{x\to-1} g(x) = \infty$, then $\lim_{x\to-1} \frac{f(x)}{g(x)} = 1$. False

(It could be anything.)

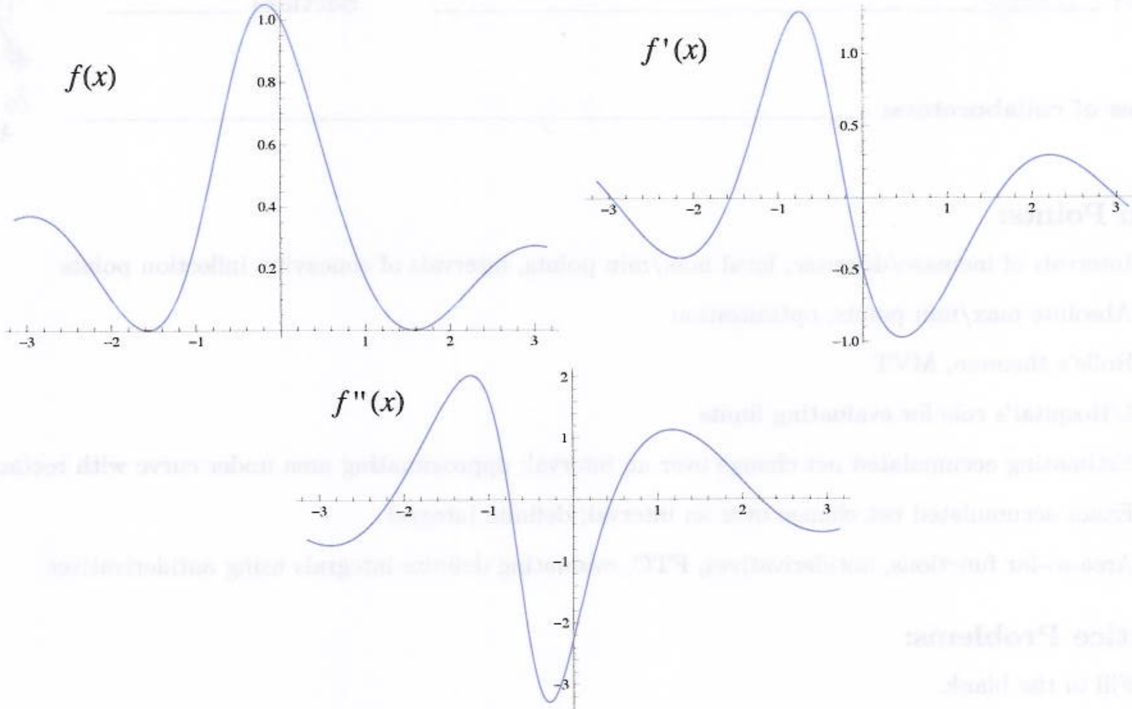
If $\int_5^{10} f(x)\,dx = 7$ and $\int_0^{10} f(x)dx = 3$, then $\int_0^5 f(x)\,dx =$ −4

Typo

If $g(x) = \int_1^x \arctan(x)\,dx$ then $A'(x) =$ arctanx.

True or False: If $F'(x) = G'(x)$, then $F(x) = G(x)$. False

(In general, F(x) & G(x) differ by a constant.)



2. Graphs of f, f', and f'' are shown above.

   (a) Estimate the x-coordinates of the local maximum points of f. Your answer should be within 0.2 units of the actual value.

   $x = -3, -0.2, 3$

   f' + −

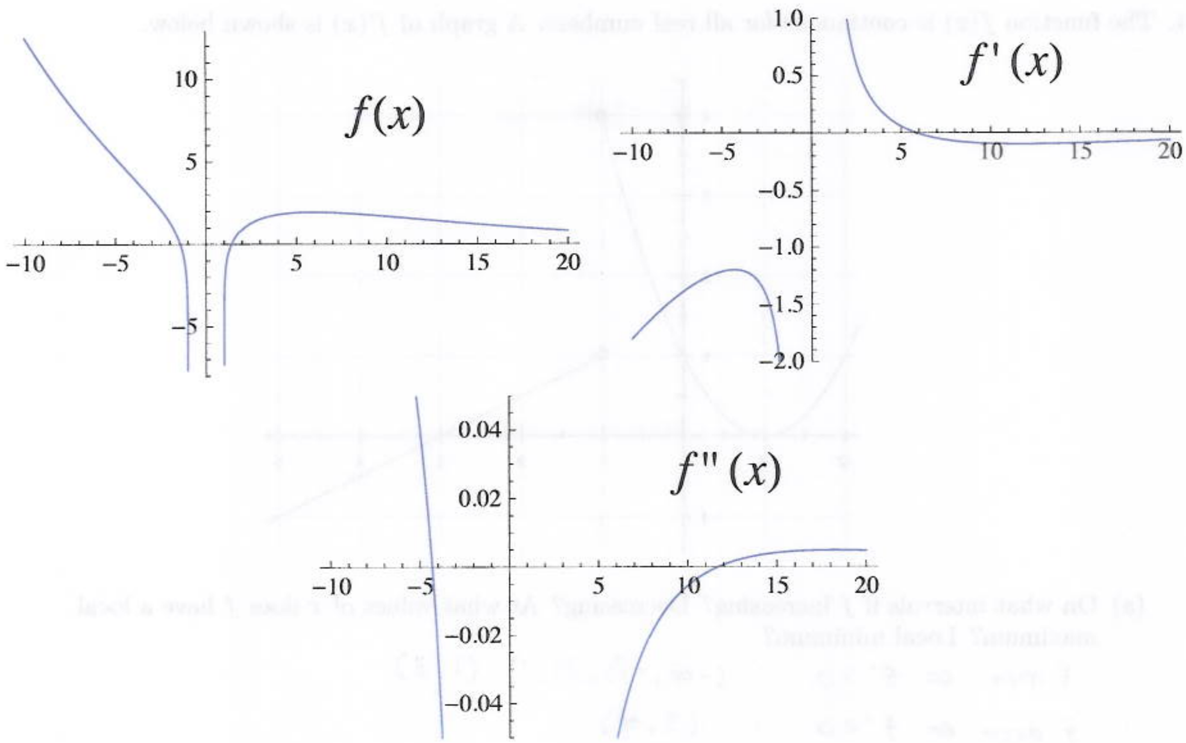
   (b) Estimate the x-coordinates of the local minimum points of f. Your answer should be within 0.2 units of the actual value.

   $x = -1.6, 1.6$

   f' − +

   (c) Estimate the x-coordinates of the inflection points of f. Your answer should be within 0.2 units of the actual value.

   $x = -2.2, -0.8, 0.4, 2.2$



3. Graphs of f, f', and f'' are shown above.

(a) Estimate the x-coordinates of any local maximum points of f. Your answer(s) should be within 1 unit of the actual values. ("Round to the nearest tick mark.")

$x = 6$

f' + − & number in domain of f

(b) Estimate the x-coordinates of any local minimum points of f. Your answer(s) should be within 1 unit of the actual values.

none

f' − +

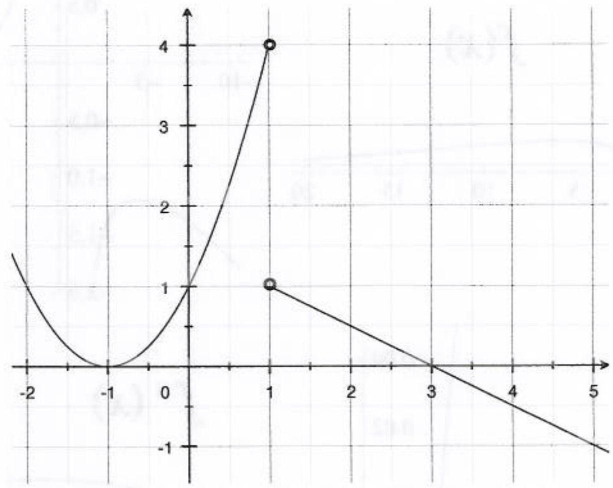
(c) Estimate the x-coordinates of any inflection points of f. Your answer(s) should be within 1 unit of the actual values.

$x = -4, 12$

$f'' = 0$ & number in domain of f

& +/- or -/+

4. The function $f(x)$ is continuous for all real numbers. A graph of $f'(x)$ is shown below.



(a) On what intervals if f increasing? Decreasing? At what values of x does f have a local maximum? Local minimum?

f incr $\Leftarrow f' > 0$: $(-\infty, -1), (-1, 1), (1, 3)$

f decr $\Leftarrow f' < 0$: $(3, \infty)$

$+ \quad + \quad + \quad -$; $-1 \quad 1 \quad 3$

local max at $x = 3$

no local min

(b) On what intervals is f concave up? Concave down? At what values of x does f have an inflection point?

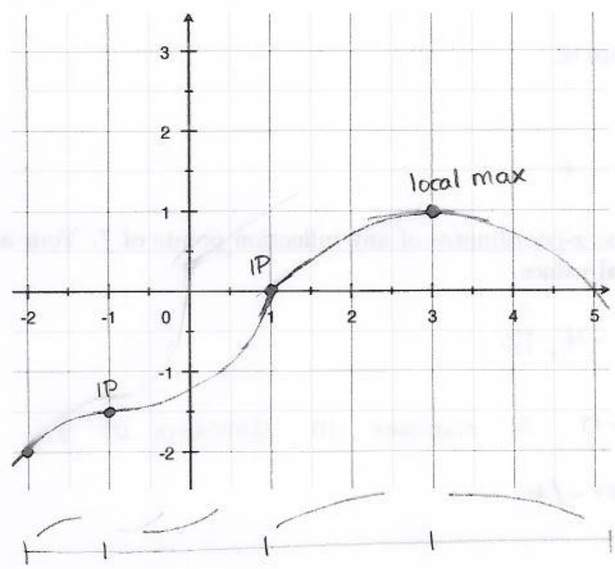
f c.u $\Leftarrow f'$ incr : $(-1, 1)$

f c.d. $\Leftarrow f'$ decr : $(-\infty, -1), (1, \infty)$

$- \quad + \quad -$; $-1 \quad 1$

IP at $x = -1$ & $x = 1$

(c) Assuming that $f(-2) = -2$, sketch a graph of f on the set of axes below.



→ Change to $[0,1]$ ✓

5. Find the local and absolute maximum and minimum values (y-values) of $f(x) = x\sqrt{1-x}$ on the interval $[-1,1]$, and state the locations (x-values) at which they occur.

$$f(x) = x(1-x)^{1/2}$$

$$f'(x) = (1-x)^{1/2} + x\left(\tfrac{1}{2}(1-x)^{-1/2}\right)(-1)$$

$$= \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$f'(x) = 0$: $x = 2/3$ $\qquad$ $f'(x)$ DNE : $x = 1$

$f(-1) = -\sqrt{2}$

$f(\frac{2}{3}) = \frac{2}{3}\sqrt{\frac{1}{3}}$

$f(1) = 0$

$f(0) = 0$

abs max value : $\frac{2}{3\sqrt{3}}$ at $x = 2/3$

abs min value : $-\sqrt{2}$ at $x = -1$

↳ 0 at $x = 0, 1$

6. Find the absolute maximum and minimum values (y-values) of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1,1]$, and state the locations (x-values) at which they occur.

$$f'(x) = 3x^2 - 6x = 3x(x^2-2) = 3x(x-\sqrt{2})(x+\sqrt{2})$$

$f'(x) = 0$ at $x = 0, -\sqrt{2}, +\sqrt{2}$

Crit #s in domain : $x = 0$

$f(-1) = -1-3+1 = -3$

$f(0) = 1$

$f(1) = 1-3+1 = -1$

abs max value : 1 at $x = 0$

abs min value : -3 at $x = -1$

✓

7. The energy expended by a bird per day, E, depends on the time spent foraging for food per dsay, F hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense. Find the foraging time that minimizes energy expenditure if

Domain $(0, 24]$

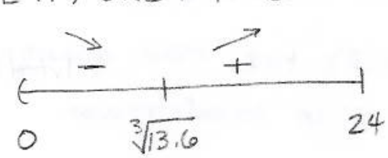
$$E = 0.25F + \frac{1.7}{F^2}$$

$$E'(F) = 0.25 - \frac{2(1.7)}{F^3} = 0.25 - \frac{3.4}{F^3}$$

$E'(F) = 0$: $\frac{3.4}{F^3} = 0.25$: $F^3 = \frac{3.4}{0.25} = 12 + 1.6 = 13.6$

$F = \sqrt[3]{13.6}$

$E'(F)$ DNE : $F = 0$



A bird will minimize energy expenditure if it forages for $\sqrt[3]{13.6}$ hrs per day.

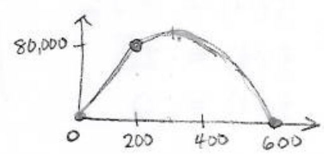
8. An apple tree produces, on average, 400 kg of fruit each season. However, if more than 200 trees are planted per km^2, crowding reduces the yield by 1kg for each tree over 200.

   (a) Express the total yield from one square kilometer as a function of the number of trees on it. Graph this function.

   n = # trees planted on 1 km^2
   y = total yield of trees planted on 1 km^2

   If $n < 200$, $y = 400n$
   If $n \geq 200$, per tree yield is $400 - (n-200) = 600 - n$
   so total yield is $y = (600-n)n$

   $$\Rightarrow y(n) = \begin{cases} 400n & \text{if } n < 200 \\ (600-n)\,n & \text{if } n \geq 200 \end{cases}$$



   (b) How many trees should a farmer plant on each square kilometer to maximize yield?

   max yield for $n \leq 200$ is $y(200) = 80{,}000$
   For $n > 200$:
   $y'(n) = 600 - 2n = 2(300-n)$

   $+ \quad -$
   200 300 600
   $\Rightarrow$ max yield at $n = 300$

   Farmer should plant 300 trees to maximize yield.

9. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given, in mg, by

$$q(t) = 20(e^{-t} - e^{-2t})$$

   (a) How much of the drug is in the bloodstream at time $t = 0$?

   $q(0) = 20(1-1) = 0$

   (b) When is the maximum quantity of drug in the bloodstream? What is that maximum?

   $q'(t) = 20(-e^{-t} + 2e^{-2t}) = 20(e^{-2t} - e^{-t}) = 20e^{-t}(2e^{-t} - 1)$
   $q'(t) = 0$ when $e^{-t} = 1/2$, i.e. $t = \ln(2)$

   $+ \quad -$
   0 $\ln(2)$

   max occurs at $t = \ln(2)$
   $q(\ln 2) = 20(1/2 - 1/4) = 5$

   After $\ln(2)$ hrs, max quantity (5 mg) of drug is in bloodstream.

   (c) In the long run, what happens to the quantity?

   $\lim_{t \to \infty} q(t) = 20(0-0) = 0$

   In the long run, the quantity of drug in the bloodstream approaches zero.

10. Evaluate the following limits, justifying your work carefully. Interpret your answers in terms of rates of growth or decay.

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right)}{2x} = \lim_{x\to\infty} \frac{\frac{1}{2x}}{2x} = \lim_{x\to\infty} \frac{1}{4x^2} = 0$

"$\frac{\infty}{\infty}$"

$\Rightarrow$ x^2 grows faster than $\ln\sqrt{x}$ as $x\to\infty$

(b) $\displaystyle\lim_{x\to0^+} \frac{\cos x}{\sqrt[4]{x^5}} = +\infty$

"$\frac{1}{0}$"

	size	sign
top	≈ 1	+
bottom	≈ 0	+

As $x\to 0^+$, the top approaches 1, while the bottom $\to 0$ & is positive. So the quotient increases infinitely.

$\Rightarrow$ $\cos x$ approaches a finite number while $\sqrt[4]{x^5}$ decays to zero

(c) $\displaystyle\lim_{x\to0^+} \sin x \ln x = \lim_{x\to0^+} \frac{\ln x}{\csc x} \overset{H}{=} \lim_{x\to0^+} \frac{1/x}{-\csc x\cot x} = \lim_{x\to0^+} \frac{\sin x\tan x}{x}$

"$0\cdot\infty$" "$\frac{\infty}{\infty}$" "$\frac{0}{0}$"

$\displaystyle\overset{H}{=} \lim_{x\to0^+} \frac{\cos x\tan x + \sin x\sec^2 x}{1} = (1)(0) + (0)(1) = 0$

$\Rightarrow$ $\sin x$ decays to zero faster than $\ln x$ grows to $-\infty$.

11. For each function below, find the most general form of the antiderivative:

(a) $s(t) = 2\cos t + 4t^3 - t^{-1}$

$S(t) = 2\sin t + t^4 - \ln|t| + C$

Change! → (b) $V(x) = \dfrac{\cos x + 1}{\sin x}$ $\dfrac{\cos x + 1}{\sin^2 x} = \cot x \csc x + \csc^2 x$

$W(x) = -\csc x - \cot x + C$

$= -\dfrac{(1+\cos x)}{\sin x} + C$

(c) $u(z) = \sqrt{z}(z - 1/z) = z^{3/2} - z^{-1/2}$

$U(z) = \frac{2}{5} z^{5/2} - 2z^{1/2} + C$

$= \frac{2}{5}\sqrt{z}(z^2 - 5) + C$

12. Evaluate the definite integrals:

(a) $\int_0^1 y^2 - y + 1\, dy = \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + y\right)\Big|_0^1 = \left(\frac{1}{3} - \frac{1}{2} + 1\right) - 0$

$= \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$

(b) $\int_0^{\ln 4} e^x + 5\, dx = \left(e^x + 5x\right)\Big|_0^{\ln 4} = (4 + 5\ln 4) - (1 + 0)$

$= 3 + 5\ln 4$

(c) $\int_{-\pi}^{\pi} 2\sin x + \cos x\, dx = (-2\cos x + \sin x)\Big|_{-\pi}^{\pi}$

$= (-2\cos\pi + \sin\pi) - (-2\cos(-\pi) + \sin(-\pi))$

$= (2+0) - (2+0)$

$= 0$

periodic!

13. Find the area under the curve $y = 1/x^2$ from $x = 1$ to $x = 3$.

$$\int_1^3 \frac{1}{x^2}\,dx = \int_1^3 x^{-2}\,dx = (-1)x^{-1}\Big|_1^3$$

$$= \left(-\frac{1}{3}\right) - \left(-\frac{1}{1}\right) = -\frac{1}{3} + 1 = \frac{2}{3}$$

14. George steadily accelerates from a stop light and records his velocity (m/s) every three seconds, resulting in the following table:

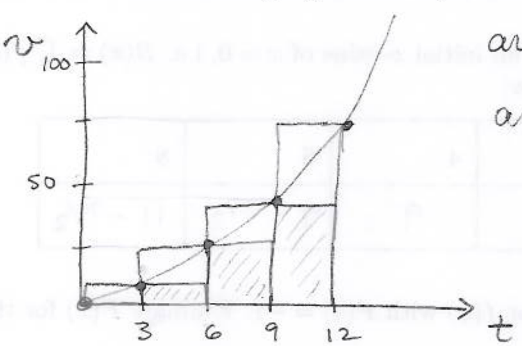
time (s)	0	3	6	9	12
velocity (m/s)	0	10	25	45	75

(a) Give lower and upper estimates for the distance George traveled in the first 12 seconds after leaving the stop light.

$\Delta s \approx 3(10 + 25 + 45 + 75) = 3(155) = 450 + 15 = 465$ m (upper)

$\Delta s \approx 3(0 + 10 + 25 + 45) = 3(80) = 240$ m (lower)

(b) On a sketch of velocity against time, show the lower and upper estimates of part (a).



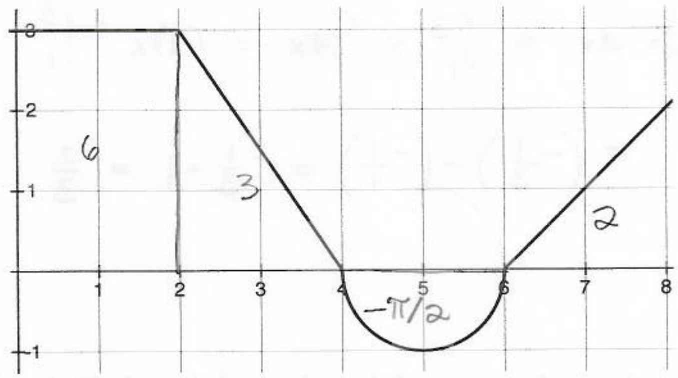
area of rectangles above curve $= 465$
area of rectangles below curve $= 240$

(c) What is your best estimate for the distance George traveled in these 12 seconds?

avg: $(465 + 240)/2 = \frac{705}{2} = 350 + 2.5 = 352.5$

Best est: $\Delta s \approx 352.5$ m

15. Consider the function $f(x)$ below.



(a) Evaluate the integrals:

$$\int_0^4 f(x)\,dx = 9$$

$$\int_4^5 f(x)\,dx = -\pi/4$$

$$\int_6^8 2f(x)+1\,dx = 2\int_6^8 f(x)\,dx + \int_6^8 1\,dx = 2(2)+2 = 6$$

(b) Let $A(x)$ be the area-so-far function with an initial x-value of $x=4$, i.e. $A(x)=\int_4^x f(x)\,dx$. Evaluate $A(x)$ for the x-values given below:

x	0	2	4	6	8
$A(x)$	-9	-3	0	$-\pi/2$	$2-\pi/2$

(c) Let $B(x)$ be the area-so-far function with an initial x-value of $x=0$, i.e. $B(x)=\int_0^x f(x)\,dx$. Evaluate $B(x)$ for the x-values given below:

x	0	2	4	6	8
$B(x)$	0	6	9	$9-\pi/2$	$11-\pi/2$

(d) Let $F(x)$ be a continuous antiderivative for $f(x)$ with $F(4)=-1$. Evaluate $F(x)$ for the x-values given below:

x	0	2	4	6	8
$F(x)$	-10	-4	-1	$-1-\pi/2$	$1-\pi/2$