Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. differentiating and integrating power series
2. power series representations for more functions

1. Differentiating and Integrating Power Series

Recall that we can consider a power series as polynomial with an infinite number of terms. For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

and

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} + \ldots + \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \ldots$$

One might hope that, to differentiate (or integrate) a power series, we can simply differentiate (or integrate) like polynomials, and, in fact, this is true! This is one of the main reasons power series are such a useful way to represent a function.

One also might be concerned (and rightly so!) about the convergence of a power series obtained by differentiation (or integration). It turns out that the radius of convergence does not change, but that the behavior at the endpoints of the interval of convergence may change.

Exercises.

1. In this exercise we differentiate and integrate $f(x) = \sum_{n=0}^{\infty} x^n$.

   (a) Write out $f(x)$ as a polynomial with infinitely many terms, then differentiate term by term to find $f'(x)$.

   (b) Write the derivative $f'(x)$ in sigma notation.

(c) Again, write out $f(x)$ as a polynomial with infinitely many terms, and, this time, integrate term by term to find the antiderivatives of $f(x)$.

(d) Write the antiderivatives of f in sigma notation.

2. In this exercise we differentiate and integrate $g(x) = \sum_{n=0}^{\infty} 3^n x^{2n}$.

(a) Write out $g(x)$ as a polynomial with infinitely many terms, then differentiate term by term to find $g'(x)$.

(b) Write the derivative $g'(x)$ in sigma notation.

(c) Again, write out $g(x)$ as a polynomial with infinitely many terms, and, this time, integrate term by term to find the antiderivatives of $g(x)$.

(d) Write the antiderivatives of g in sigma notation.

2. Finding More Power Series Representations

We can now use differentiation and integration to allow us to find power series representations for more functions. See Examples 5-7 in the textbook.

Exercises.

3. Recall that the power series in Problem 1 can also be written as $f(x) = \dfrac{1}{1-x}$ for $|x| < 1$.

   (a) Find the derivative of $f(x) = \dfrac{1}{1-x}$.

   (b) Find $\int f(x)dx$.

   (c) Find a power series representation for $\dfrac{1}{(1-x)^2}$. (Hint: Use Problem 1.) What is the radius of convergence? What is the interval of convergence?

(d) Find a power series representation for $-\ln(1-x)$. What is the radius of convergence? What is the integral of convergence?

4. In this exercise we return to the power series in Problem 2, $g(x) = \sum_{n=0}^{\infty} 3^n x^{2n}$.

(a) Use the formula for the sum of a geometric series to write $g(x)$ as a rational function.

(b) Find the derivative of the rational function $g(x)$.

(c) Find a power series representation for $\dfrac{6x}{(1-3x^2)^2}$. What is the radius of convergence? The interval of convergence?

5. Find a power series representation for $h(x) = \ln(3 + x)$. Find the radius and interval of convergence.

6. (a) Find a power series representation for $p(x) = \arctan(x)$. (Hint: Try taking the derivative of $p(x)$ first. Then write $p'(x)$ as a power series. Then antidifferentiate to find a power series representation for $p(x)$.)

   (b) Find a power series representation for $q(x) = x^2 \arctan(x^3)$.