Math 114, Prep 7.4 (Partial Fractions)

Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Finding simple partial fractions decompositions by hand
2. Using *Mathematica* to find more complicated partial fractions decompositions
3. Using partial fractions decompositions to simplify integration

1. Finding Simple Partial Fractions Decompositions

Partial fractions is an algebraic technique that can be helpful for integration. In particular, the partial fractions decomposition is a way to rewrite a rational function as a sum of simpler rational functions, as long as the degree of the numerator is smaller than the degree of the denominator. (If the degree of the denominator is larger than the degree of the numerator, long division of polynomials can be used first.) It is a reverse process to adding rational functions, and as such requires "undoing the common denominator."

1. We will find the partial fractions decomposition for $\frac{2x+3}{x^2-4x-5}$. (See Example 2, page 486.)

   The denomniator factors as $(x+1)(x-5)$, and the partial fractions decomposition is of the form

   $$\frac{2x+3}{x^2-4x-5} = \frac{A}{x+1} + \frac{B}{x-5}$$

   (a) Clear denominators by multiplying both sides by $(x+1)(x-5)$.

   (b) Plug in $x=-1$ and solve for A.

   (c) Plug in $x=5$ and solve for B.

   (d) There is another way of finding A and B which takes longer, but is more reliable in general. It is called "the method of undetermined coefficients."

   Go back to your equation in (a), expand/foil the right hand side, and collect terms.

You should get: $(A+B)x \;+\; (B-5A)$. This means that

$$2x+3 \;=\; (A+B)x \;+\; (B-5A)$$

The only way for this to be true is if $2 = A+B$ and $3 = B-5A$. Use these two equations to solve for A and B. Double-check your answers by comparing to (b) and (c).

2. We will find the partial fractions decomposition for $\dfrac{x-9}{(x+5)(x-2)^2}$.

   Since there is a repeated linear factor, the partial fractions decomposition is of the form:

$$\frac{x-9}{(x+5)(x-2)^2} \;=\; \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

   (a) Clear denominators by multiplying both sides by $(x+5)(x-2)^2$.

   (b) Solve for A and C by plugging in specific numbers, as in 2(b) and 2(c). Can you solve for B?

   (c) Use the method of undetermined coefficients to find B. (You can use what you already know about A and C.)

2. Partial Fractions Decompositions Involving Irreductible Quadratic Factors

So far we have dealt with rational functions whose denominators factor into linear factors, but we know that this is not always going to be the case, because some polynomials, e.g. $x^2 + 1$, do not factor into linear factors. (Using the quadratic formula gives complex roots.)

It is true, however, that all polynomials factor into linear factors and irreducible quadratic factors. (This fact is not trivial, but we will take it for granted.)

For example, suppose we wished to find the partial fractions decomposition for $\dfrac{x-9}{(x+5)(x^2+1)}$. Notice that $(x^2 + 1)$ cannot be factored; it is called an **irreducible quadratic** factor. Instead of having just a number like B in the numerator, we need something of the form $Bx + C$. The partial fractions decomposition is of the form:

$$\frac{x-9}{(x+5)(x^2+1)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+1}$$

We deal with powers of irreducible quadratics analogously to powers of linear factors. For example,

$$\frac{x-9}{(x+5)(x-2)^2(x^2+1)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

is the correct form of the partial fractions decomposition of the rational function on the left side.

The `Apart` command in *Mathematica* can be used to find partial fractions decompositions. For example, to find the partial fractions decomposition in the first exercise we could have typed:

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Apart[(2x+3)/(x^2 - 4x - 5)]
```

Exercises

3. Use the `Apart` command in *Mathematica* to find the coefficients A, B, C, ... in the partial fractions decompositions below.

   (a) $\dfrac{x-9}{(x+5)(x^2+1)} = \dfrac{A}{x+5} + \dfrac{Bx+C}{x^2+1}$

   (b) $\dfrac{x-9}{(x+5)(x-2)^2(x^2+1)^2} = \dfrac{A}{x+5} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} + \dfrac{Dx+E}{x^2+1} + \dfrac{Fx+G}{(x^2+1)^2}$

3. Using the Partial Fractions Decomposition

We use the partial fractions to rewrite rational integrands as sums of simpler rational functions. It may be necessary to use substitution and/or trig substitution. Recall antiderivatives of basic rational functions:

$$\int \frac{dx}{x} = \ln|x| + C \qquad \int \frac{dx}{x^p} = \frac{1}{(1-p)\,x^{p-1}} + C \quad (p > 1) \qquad \int \frac{dx}{1+x^2} = \arctan(x) + C$$

Exercises.

4. Evaluate the integrals. Use the `Apart` command to find the partial fractions decompositions.

   (a) $\displaystyle\int \frac{10}{5x^2 - 2x^3}\,dx.$

   (b) $\displaystyle\int \frac{x^5 + 4}{x^4 + 4x^2}\,dx$