Math 114, Prep 7.8B (The Comparison Theorem)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. An indirect method for determining whether an improper integral converges
2. A careful argument: set-up, check hypotheses, apply theorem, draw conclusion

1. Introduction

Suppose we wish to determine whether or not $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges. Recall that, if it converges, it is:

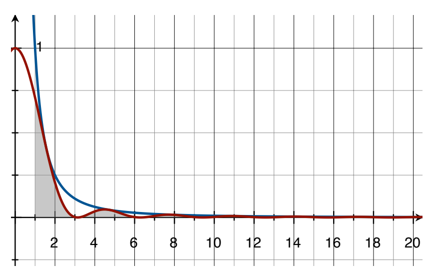
$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx = \lim_{T\to\infty}\int_1^T \frac{\sin^2(x)}{x^2}\,dx$$

If we had an antiderivative for $\frac{\sin^2(x)}{x^2}$, we could use the FTC, and then evaluate the limit, as in the problems discussed last time. However, it is not at all obvious how to find an antiderivative in this case, so we take an alternate route: we compare to a function whose antiderivative we do know: $\frac{1}{x^2}$.

We use the fact that $\sin^2 x$ is bounded between 0 and 1 to compare $f(x)$ and $g(x)$:

$$0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \ \le = \ \frac{\sin^2(x)}{x^2} \ \le = \ \frac{1}{x^2}$$

Thus is is reasonable to conclude that the area under the curve $y = \frac{\sin^2(x)}{x^2}$ is less than the area under the curve $y = \frac{1}{x^2}$, as is illustrated in the graph below.



The fact that that $\int_1^\infty \frac{1}{x^2}\,dx$ is finite implies that $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is finite, i.e. the integral converges. This is the idea behind the Comparison Theorem.

Exercises.

1. Do you think that $\int_1^\infty \frac{\cos(x)+1}{2x^3}\,dx$ converges or diverges? Explain your reasoning. (Hint: Use the fact that $-1 \le \cos(x) \le 1$. Look at a graph if you need to.)

2. Do you think that $\int_1^\infty \frac{1+e^{-x}}{\sqrt{x}}\,dx$ converges or diverges? Explain your reasoning.

2. Using the Comparison Theorem

To justify the guesses we have made above, we need to use the Comparison Theorem. Whenever you want to use a theorem to draw a conclusion, you need to: (1) set up the theorem, (2) check that the hypotheses of the theorem are satisfied, (3) cite the theorem and draw your conclusion.

Exercises

3. State the Comparison Theorem (given at the bottom of p 525.)

4. In this problem, you will give a careful argument justifying our guess that $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges.

   (a) (Set-up.) There are three things to specify: f, g, and a. State what each is in this example.

   (b) (Check hypotheses.) There are several things to check: (1) that f and g are continuous functions, (2) that $f(x) \geq g(x) \geq 0$ for all $x \geq a$, and (3) that $\int_a^\infty f(x)dx$ is convergent. Explain why all of these are true in this example.

(c) (Cite theorem and draw conclusion.) Here you merely need to say, "By the Comparison Theorem, we can conclude that ..." (Finish this sentence.)

5. Give a careful argument to justify your guess in Exercise 1. (Use the outline given in Exercise 4.)

6. Give a careful argument to justify your guess in Exercise 2. (Use the outline given in Exercise 4.)