Math 114, Prep 11.3 (Integral Test)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Bring your laptops to class, so we can use *Mathematica*!

Main Points:

1. the integral test (check hypotheses!) and the p-test
2. estimating the sum of a series

1. Comparing Infinite Series with Improper Integrals

Given that we used Riemann sums to approximate integrals, it should be no surprise that we can use improper integrals to understand infinite series. Read pages 714-716, up to and including Example 1.

Exercises.

1. Read the Integral Test (in the red box, page 716).

   (a) State the integral test below.

   (b) In order to use the integral test, we need the function $f(x)$ to satisfy three conditions. What are the three conditions?

   (c) Read the Note after the Integral Test, and explain what it says in your own words.

2. In this exercise, we will use the Integral Test to determine whether the series $\sum_{i=1}^{\infty} e^{-n}$ converges.

   (a) (**Set-up**) Define a function $f(x)$ that "matches" the terms of the series, i.e. $f(x)$ should be a function such that $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$, etc.

   $f(x) \quad =$

   (b) (**Check Hypotheses**) Check to make sure the function $f(x)$ satisfies the three conditions stated in the Integral Test. (See 1b.)

   (c) (**Improper Integral**) Determine whether the improper integral $\int_1^\infty f(x)\,dx$ converges or diverges. (Hint: Remember that $\int_1^\infty f(x)\,dx \;=\; \lim_{T\to\infty} \int_1^T f(x)\,dx$.)

   (d) (**Apply Test and Draw Conclusion**) Use the Integral Test to determine whether the series converges, and state your conclusion in a sentence. Make sure to include the phrase "by the Integral Test" somewhere in your sentence.

3. Read Example 2, and summarize the results (which are given in a red box, immediately following the example.) This result is called **the** p**-test**.

4. Use the p-test to determine whether the following series converge or diverge:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^{\pi}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(n^{-1.4} + 3n^{-1.2}\right)$

5. In this exercise, we will use the Integral Test to determine whether the series $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(n)}{n^2}$ converges.

   (a) (**Set-up**) Define a function $f(x)$ that "matches" the terms of the series, i.e. $f(x)$ should be a function such that $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$, etc.

   $f(x) \quad =$

(b) (**Check Hypotheses**) Check to make sure the function $f(x)$ satisfies the three conditions stated in the Integral Test. (Hint: To show that the function is eventually decreasing, you need to look at $f'(x)$ and show that it is negative, at least as long as x is sufficiently large.)

(c) (**Improper Integral**) Use integration by parts to find an antiderivative for $f(x)$, and then determine whether the improper integral $\int_1^\infty f(x)\,dx$ converges or diverges. (Hint: In the last step, when you are evaluating the limit as $T \to \infty$, you will need to use L'Hospital's Rule.)

(d) (**Apply Test and Draw Conclusion**) Use the Integral Test to determine whether the series converges, and state your conclusion in a sentence. Make sure to include the phrase "by the Integral Test" somewhere in your sentence.

2. Estimating the Sum of a Series

So far we have only used the improper integrals to determine whether the corresponding series converges or diverges. We can do better than this! We can use improper integrals to estimate the sum of the corresponding series. Read pages 718 and 719.

Exercises.

6. (a) State the Remainder Estimate for the Integral Test.

   (b) We can use the remainder estimates to derive a lower bound and an upper bound for the sum of a (convergent) series. See the red box labeled [3], and state the bounds given there.

7. (See Examples 5 and 6.) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$. Euler showed that the sum of this series is $\frac{\pi^4}{90}$.

   (a) Use *Mathematica* to find the partial sum $S_{10} = \displaystyle\sum_{n=1}^{10} \frac{1}{n^4}$. (Keep at least six digits.)

   ```
   N[Sum[1/n^4, {n, 1, 10}]]
   ```

   (b) We could use S_{10} to approximate the infinite series. Estimate the error in using S_{10} as an approximation by bounding the remainder R_{10}.

(c) (**Challenge**) Use the inequality [3] on page 719, with $n = 10$, to give an improved estimate of the sum. Compare your estimate with the exact value found by Euler.

(d) (**Challenge**) Find a value of n so that S_n is within 0.00001 of the sum.