Math 114, Prep 11.10A (Intro to Taylor Series)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. Taylor series and Taylor polynomials
2. using Taylor series

Recall that we can consider a power series as polynomial with an infinite number of terms. For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

We can also consider this as representing a *family of polynomial approximations*:

$$\begin{aligned}
\frac{1}{1-x} &\approx s_0(x) = 1 \\
\frac{1}{1-x} &\approx s_1(x) = 1 + x \\
\frac{1}{1-x} &\approx s_2(x) = 1 + x + x^2 \\
&\vdots \\
\frac{1}{1-x} &\approx s_n(x) = 1 + x + x^2 + x^3 + \ldots + x^n \\
&\vdots
\end{aligned}$$

Recall (from Calc I), that we have another way of obtaining polynomial approximations: via derivatives. If a function $f(x)$ is continuous at $x = a$, we can approximate $f(x)$ by a constant function $f(x) \approx f(a)$, at least near $x = a$. If $f(x)$ is differentiable, we can approximate it by a linear function (the tangent line).

$$\begin{aligned}
f(x) &\approx f(a) \\
f(x) &\approx f(a) + f'(a)(x-a)
\end{aligned}$$

Extending this idea leads to the notion of Taylor polynomials and Taylor series. A Taylor series centered around $x = 0$ is sometimes called a Maclaurin series. (See Example 1.)

Exercises.

1. Read pages 753-755 of the section. State Theorem 5 (page 754).

2. Read Examples 4 and 5.

   (a) What is the Taylor series for $\sin(x)$ centered at $x = 0$? For $\cos(x)$, centered at $x = 0$?

   (b) What are the first three distinct nonzero Taylor polynomials for $\sin x$ at $x = 0$? For $\cos x$?

   Note: For $\sin x$, the first three distinct nonzero Taylor polynomials are $T_1(x)$, $T_3(x)$, and $T_5(x)$, and for $\cos x$ they are $T_0(x)$, $T_2(x)$ and $T_4(x)$.

3. Use Theorem 5 to find the first three distinct nonzero Taylor polynomials for $f(x) = \sin(2x)$ centered at $x = 0$.

4. (a) Use Theorem 5 to find the first four Taylor polynomials of $g(x) = \frac{1}{x}$ centered at $x = -3$.

   (b) Find the Taylor series of $g(x)$ centered at $x = -3$. What is the radius of convergence?

5. We can use known Taylor series to find Taylor series for related functions. (See Example 6.)

   (a) Use the Taylor series for $\ln(1 + x)$ to find a Taylor series for $x \ln(1 + x)$.

(b) Use the Taylor series for e^x to find a Taylor series for $\dfrac{e^x - 1}{x}$.

Hint. Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, subtracting 1 gives: $e^x - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

(c) Use the Taylor series for $\cos(x)$ to find a Taylor series for $\cos(x^2)$.

Hint. Substitute $u = x^2$ in for x in the Taylor series for $\cos x$.

(d) Use the Taylor series for e^x to find a Taylor series for $e^{-x/2}$.