Practice Problems for 11.9 and 11.10

1. Find power series representations for each of the following functions and state the radius of convergence. Write your answer in two ways: (i) with sigma notation: $\sum_{n=0}^{\infty} c_n\, x^n$ and (ii) by writing out the first four terms followed by ellipses: $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$.

   (a) $f(x) \;=\; \dfrac{1}{2+x}$

   Use the geometric series with $a = 1/2$ and $r = -x/2$.

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}\, x^n \;=\; \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots$$

   Since a geometric series converges when $|r| < 1$ and diverges otherwise, this power series converges when $|-x/2| < 1$, i.e. $|x| < 2$, and diverges otherwise, so the radius of convergence is $R = 2$.

   (b) $g(x) \;=\; \dfrac{1}{(2+x)^2}$

   Notice that $g(x) = -f'(x)$, differentiate the power series above, and shift the index.

$$g(x) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}}\, x^n \;=\; \frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16} - \frac{x^3}{8} + \ldots$$

   Since the radius of convergece of a power series does not change when differentiated, the radius of convergence of this series is $R = 2$.

   (c) $h(x) \;=\; \dfrac{x^2}{(2+x)^3}$

   Notice that $h(x) = (-x^2/2)\, g'(x)$, differentiate the power series above and multiply by $-x^2/2$.

$$h(x) \;=\; \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{2^{n+2}}\, x^n \;=\; \frac{x^2}{8} - \frac{3x^3}{16} + \frac{3x^4}{16} - \frac{5x^5}{32} + \ldots$$

   Since the radius of convergece of a power series does not change when differentiated, the radius of convergence of this series is $R = 2$.

   (d) $r(x) \;=\; \ln(2+x)$

   Notice that $r(x) = f'(x)$, **antidifferentiate** the power series for $f(x)$, shift the index, and solve for the constant by plugging in $x = 0$.

$$r(x) \;=\; \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n}\, x^n \;=\; \ln(2) + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \ldots$$

   Since the radius of convergece of a power series does not change when antidifferentiated, the radius of convergence of this series is $R = 2$.

2. Find the fourth-degree Taylor polynomial centered at $x = a$ for the function $f(x)$ where

   (a) $f(x) = \sin(x)$, $a = \pi/2$

   The fourth-degree Taylor polynomial is: $T_4(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4$.

   (b) $f(x) = \ln(x)$, $a = 1$

   $T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

   (c) $f(x) = \dfrac{1}{2+x}$, $a = 1$

   $T_4(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \frac{1}{81}(x-1)^3 + \frac{1}{243}(x-1)^4$

3. Evaluate the indefinite integral as a power series. What is the radius of convergence?

   (a) $\displaystyle\int \frac{t}{1-t^8}\,dt$

   Since $\displaystyle\frac{t}{1-t^8} = \sum_{n=0}^{\infty} t^{8n+1}$ with radius of convergence $R = 1$,

   $\displaystyle\int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$ with radius of convergence $R = 1$.

   (b) $\displaystyle\int \frac{t}{1+t^3}\,dt$

   Since $\displaystyle\frac{t}{1+t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}$ with radius of convergence $R = 1$,

   $\displaystyle\int \frac{t}{1+t^3}\,dt = C + \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n+2}}{3n+2}$ with radius of convergence $R = 1$.

   (c) $\int e^{-x^2}\,dx$

   Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with $R = \infty$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ with $R = \infty$, so

   $$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

4. (**Challenge**) Estimate $\int e^{-x^2}\,dx$ accurate to $1/10$.

   Using the antiderivative found above,

   $$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} = 1 - \frac{1}{3} + \frac{1}{10} - \ldots$$

   By the Alternating Series Estimation Theorem, the partial sum estimate $1 - \frac{1}{3}$ is accurate to 1/10. Thus $\int_0^1 e^{-x^2}\,dx \approx 2/3$ is accurate to 1/10.