Math 114-01/03, Calculus II, Fall 2017

Section 01: MWF 8:15-9:20, OSS 333   Section 03: MWF 10:55-12:00 OSS 226

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Course Prerequisites: Successful completion (C- or better) of trigonometry based Calculus I (Math 113), or its equivalent.

Credits and Workload Expectations: 4 credits: 8-10 hours per week outside the classroom.

Course Materials and Resourses:

- Textbook: *Calculus: Single and Multivariable*, 7th ed., Hughes-Hallett, McCallum, et. al.

- *Mathematica* Software: Request a free download under St. Thomas' site license. https://www.stthomas.edu/its/students/personallyownedcomputers/software/

- Math Resource Center (MaRC, OSS 235) : free drop-in peer tutoring, group study areas, solution manuals, *Mathematica* help, ...

Course Objectives:

- Gaining a basic understanding of the subject (e.g. factual knowledge, methods, principles, generalizations, and theories)

- Learning to apply course material (to improve thinking, problem solving, and decisions)

- Developing specific skills, competencies, and points of view needed by professionals in the field most closely related to this course

- Developing skill in expressing oneself orally or in writing

Assignments: To prepare for class you will be assigned reading, along with **reading questions** and **discussion problems** related to the reading. You will also be assigned a **problem set** consisting of the discussion problems and some additional problems related to material that we have already discussed in class. For each topic, you will write up one **quality solution**. Occasionally, in-class participation or presentations will be graded.

Collaboration and consultation: I encourage you, when working on homework, to collaborate with fellow students, to reread the textbook, and to ask the professor or the MaRC tutors for help. You are also free to consult other textbooks or online resources for general information on the topic. However, you may not consult any worked solution to an assigned homework problem. This includes but is not limited to: the student solution manual, the instructor solution manual, WileyPlus, other students' written homework, and any online solution. Moreover, when you are writing up your homework, you must write your own solutions, in your own words. If in doubt about the acceptability of a certain kind of collaboration or consultation, ask the professor.

Late Work: Late work is typically not accepted. The lowest three scores in each assignment category will be dropped at the end of the semester. Extensions may be granted if requested before the due date, and work may certainly be submitted before the due date, if arrangements have been made with the professor in advance. If there is a serious, unforeseeable reason for missing more than three days of class, it is the student's responsibility to contact the professor as soon as possible and to make appointments with the professor and with Academic Counseling upon returning to classes to make a plan for making up missed work.

Missed Exams: Make-up midterm exams may be given to students with legitimate excuses such as serious illness, university sponsored events, etc., as long as the make-up exam can be taken within a reasonable time frame. If it is not possible to schedule a make-up exam within a reasonable time frame, the grade for the midterm may be prorated from the final exam. Written documentation may be required. Rescheduling the final is not possible except under very extreme circumstances.

Incompletes: Grades of I are normally not given in this course. However, they may be granted due to extenuating circumstances especially if (i) the majority of the course work has been completed at a level of C or better and (ii) the student demonstrates the ability to complete the remaining coursework outside of the classroom. In such cases, a well-documented petition should be submitted to the professor before the last day of classes. Please see the university policies on withdrawals and incomplete grades.

Final Course Grade: The overall score for this course will be computed as outlined below. Final letter grades will be assigned based on the overall score, with the two major components, written solutions and exams also being considered separately. In particular, the final letter grade will not be higher than one letter grade above the level of the work on written solutions or the work on exams. Exceptional performance on the final may also be taken into account.

- Preparation (10%): reading questions (RQ, 5%), discussion preparation (D, 5%)
- Mastery Assignments (35%): problem sets (P, 5%), and quality solutions and oral presentations (QS, 30%)
- Midterm Exams (30%): tentatively Mon Oct 2, Mon Oct 30, and Wed Nov 29
- Final Exam (20%): cumulative; **1:30-3:30 pm Thurs Dec 21**, location TBA
- Best Exam (5%): at the end of the semester the score for the best exam will contribute an extra 5% towards the overall score

Disability Accommodations: Academic accommodations will be provided for qualified students with documented disabilities including but not limited to mental health diagnoses, learning disabilities, Attention Deficit Disorder, Autism, chronic medical conditions, visual, mobility, and hearing disabilities. Students are invited to contact the Disability Resources office about accommodations early in the semester. Appointments can be made by calling 651-962-6315 or in person in Murray Herrick, room 110. For further information, you can locate the Disability Resources office on the web at http://www.stthomas.edu/enhancementprog/.

Math 114-01, F2017, Detailed Schedule

Mon	Wed	Fri
Sep 4, 2017	Sep 6, 2017	Sep 8, 2017
Labor Day	Intro to Course Due today: Read 2.4, 3.9 Next class: RQ 10.1, D 2.4/3.9	Derivatives and Approximation (2.4, 3.9) Due today: RQ 10.1, D 2.4/3.9 Next class: RQ 10.2, D 10.1, P&QS 2.4/3.9
Sep 11, 2017	Sep 13, 2017	Sep 15, 2017
Taylor Polynomials (10.1) Due today: RQ 10.2, D 10.1, P&QS 3.9 Next class: RQ 5.1/5.2, D 10.2, P&QS 10.1	Taylor Series (10.2) Due today: RQ 5.1/5.2, D 10.2, P&QS 10.1 Next class: RQ 7.5, D 5.1/5.2, P&QS 10.2	Definite Integrals and Approximation (5.1, 5.2) Due today: RQ 7.5, D 5.1/5.2, P&QS 10.2 Next class: RQ 5.3/6.1/6.4, D 7.5, P&QS 5.1/5.2
Sep 18, 2017	Sep 20, 2017	Sep 22, 2017
Numerical Integration (7.5) Due today: RQ 5.3/6.1/6.4, D 7.5, P&QS 5.1/5.2 Next class: RQ 6.3/11.1, D 5.3/6.1/6.4, P&QS 7.5	The FTC (5.3, 6.1, 6.4) Due today: RQ 6.3/11.1, D 5.3/6.1/6.4, P&QS 7.5 Next class: RQ 11.2, D 6.3/11.1, P&QS 5.3/6.1/6.4	Intro to Diff. Eq. (6.3, 11.1) Due today: RQ 11.2, D 6.3/11.1, P&QS 5.3/6.1/6.4 Next class: RQ 11.4/11.5, D 11.2, P&QS 6.3/11.1
Sep 25, 2017	Sep 27, 2017	Sep 29, 2017
Slope Fields (11.2) Due today: RQ 11.4/11.5, D 11.2, P&QS 6.3/11.1 Next class: RQ 7.1, D 11.4/11.5, P&QS 11.2	Separation of Variables and Modeling (11.4, 11.5) Due today: RQ 7.1, D 11.4/11.5, P&QS 11.2 Next class: P&QS 11.4, 11.5; D Rev	Review Session Due today: D Rev, P&QS 11.4/11.5 Next class: Study for exam
Oct 2, 2017	Oct 4, 2017	Oct 6, 2017
Exam 1 Next class: RQ 7.2, D 7.1	Integration by Substitution (7.1) Due today: RQ 7.2, D 7.1 Next class: RQ 7.4A, D 7.2, P&QS 7.1	Integration by Parts (7.2) Due today: RQ 7.4A, D 7.2, P&QS 7.1 Next class: RQ 7.4B, D 7.4A, P&QS 7.2
Oct 9, 2017	Oct 11, 2017	Oct 13, 2017
Partial Fractions (7.4A) Due today: RQ 7.4B, D 7.4A, P&QS 7.2 Next class: RQ 1.8, D 7.4B, P&QS 7.4A	Trigonometric Substitutions (7.4B) Due today: RQ 1.8, D 7.4B, P&QS 7.4A Next class: RQ 4.7, D 1.8, P&QS 7.4B	Limits (1.8) Due today: RQ 4.7, D 1.8, P&QS 7.4B Next class: RQ 7.6, D 4.7, P&QS 1.8
Oct 16, 2017	Oct 18, 2017	Oct 20, 2017
L'Hopital's Rule (4.7) Due today: RQ 7.6, D 4.7, P&QS 1.8 Next class: RQ 4.8A, D 7.6, P&QS 4.7	Improper Integrals (7.6) Due today: RQ 4.8A, D 7.6, P&QS 4.7 Next class: RQ 4.8B, D 4.8A, P&QS 7.6	Parametric Equations (4.8A) Due today: RQ 4.8B, D 4.8A, P&QS 7.6 Next class: RQ 5.4, D 4.8B, P&QS 4.8A
Oct 23, 2017	Oct 25, 2017	Oct 27, 2017
Parametric Curves: Slope, Concavity, (4.8B) and Area (supplement) Due today: RQ 5.4, D 4.8B, P&QS 4.8A Next class: P&QS 4.8B; D Rev	Review Session Due today: D Rev, P&QS 4.8B Next class: Study for exam	Fall Break
Oct 30, 2017	Nov 1, 2017	Nov 3, 2017
Exam 2 **Next class: RQ 8.1, D 5.4**	Area b/w Curves; Ave Value (5.4) Due today: RQ 8.1, D 5.4 Next class: RQ 8.2A, D 8.1, P&QS 5.4	Areas and Volumes (8.1) Due today: RQ 8.2A, D 8.1, P&QS 5.4 Next class: RQ 8.2B, D 8.2A, P&QS 8.1
Nov 6, 2017	Nov 8, 2017	Nov 10, 2017
Applications to Geometry: Volumes (8.2A) Due today: RQ 8.2B, D 8.2A, P&QS 8.1 Next class: D 8.2B, P&QS 8.2A; prepare presentation	Applications to Geometry: Arc Length (8.2B) Due today: D 8.2B, P&QS 8.2A Next class: P&QS 8.2B; prepare presentation	Presentations: Density and Center of Mass (8.4) Due today: P&QS 8.2B, 8.4 presentation QS Next class: P 8.4, prepare presentation
Nov 13, 2017	Nov 15, 2017	Nov 17, 2017
Presentations: Applications to Physics (8.5) Due today: P 8.4, 8.5 presentation QS Next class: P 8.5, prepare presentation	Presentations: Applications to Economics (8.6) Due today: P 8.5, 8.6 presentation QS Next class: RQ 9.1, P 8.6, QS 8.4-8.6	Intro to Sequences (9.1A) Due today: RQ 9.1, P 8.6, QS 8.4-8.6 Next class: RQ 9.2, D 9.1B, P&QS 9.1A
Nov 20, 2017	Nov 22, 2017	Nov 24, 2017
Convergence of Sequences (9.1B) Due today: RQ 9.2, D 9.1B, P&QS 9.1A Next class: P&QS 9.1B, D Rev	Mathematica Project	Thanksgiving Break
Nov 27, 2017	Nov 29, 2017	Dec 1, 2017
Review Session Due today: D Rev, P&QS 9.1B Next class: Study for exam	**Exam 3** Next class: RQ 9.3, D 9.2	Geometric Series (9.2) Due today: RQ 9.3, D 9.2 Next class: RQ 9.4, D 9.3, P&QS 9.2
Dec 4, 2017	Dec 6, 2017	Dec 8, 2017
Convergence of Series (9.3) Due today: RQ 9.4, D 9.3, P&QS 9.2 Next class: RQ 9.5, D 9.4, P&QS 9.3	Ratio Test (9.4) Due today: RQ 9.5, D 9.4, P&QS 9.3 Next class: RQ 10.3A, D 9.5, P&QS 9.4	Power Series (9.5) Due today: RQ 10.3A, D 9.5, P&QS 9.4 Next class: RQ 10.3B, D 10.3A, P&QS 9.5
Dec 11, 2017	Dec 13, 2017	Dec 15, 2017
Taylor Series Revisited (10.2, 10.3A) Due today: RQ 10.3B, D 10.3A, P&QS 9.5 Next class: D 10.3B, P&QS 10.3A	Using Taylor Series (10.3B) Due today: D 10.3B, P&QS 10.3A Due next class: P&QS 10.3B	Open Due today: P&QS 10.3B

General Guidelines

- All written homework is due at the *beginning* of class.
- Write your name, your section (e.g. Math 114-01, Math 200-04, etc.), the assignment (e.g. P 12.1 or QS 3.9), and the due date on the front page.
- Write your name on each page.
- Staple two neat packets: one for the problem set (P), one for your quality solution (QS).
- Neat edges! If you use notebook paper, trim the frayed edges with scissors.
- Label each problem, and circle or box each final answer.
- Write neatly. If your handwriting is illegible, use a word processor.
- At the end of the problem set (P) include an evaluation box. (See below.)
- The rule of thumb is: make it easy for the grader to give you credit for your work!

Problem Sets (P)

- write up your solutions to the discussion problems and additional problems
- evaluation criteria: clarity, completeness, and correctness
- evaluation marks: thoroughly (plus), mostly (check), significantly (check minus), not at all (ex)
- evaluation box: list the three criteria and leave spaces for the evaluation marks

Quality Solutions (QS)

- A quality solution is a written explanation of how to solve a textbook problem. Your imaginary audience is a peer who is struggling to understand the topic.
- Restate the question. (That includes copying down directions that appear at the beginning of a group of problems and any tables or graphs).
- Show all your steps, and make your reasoning clear. If your solution includes a table, graph, or drawing, make sure it is clearly labeled.
- The rule of thumb is that what you write should stand on its own as an explanation.

Rubric for Quality Solutions

- Very Nice (4): clear, correct, and complete solution of the problem and good presentation
- Right Idea (3): essentially correct, but some small gaps, lack of clarity, or poor presentation
- Good Start (2): shows partial understanding, e.g. correct start, but significant flaws or gaps
- Good Effort (1): inappropriate approach, faulty reasoning, or wrong problem
- No Attempt (0): recopy problem but do not attempt to solve it

The numbers are "messages" not points! For example, getting an all 3s and 4s (with a few more 4s) would be very good, A-level work, whereas consistently scoring 2s would mean that your work does not demonstrate sufficient understanding to move on (D-level work).

Challenge Problems. Occasionally you will be assigned challenge problems. These problems do factor into your regular grade. (So don't skip them!) Challenge problems are graded on a more lenient scale: as long as you attempt the problem, your score for the problem will be one level higher than the rubric above dictates. For example, if your work on the problem would normally merit a 3 (you have the right idea), it will get a 4, because having the right idea on a challenge problem is really very good!

Main Points:

1. The practical meaning of the derivative
2. Using the derivative for linear approximation
3. Using the second derivative (concavity) to determine whether the linear approximation is an over-estimate or an under-estimate.

Overview and Example

Recall that the derivative is an **instantaneous rate of change**. Graphically, this instantaneous rate of change is represented as the **slope of a tangent**. We can use an instantaneous rate of change of a quantity to make predictions about how much the quantity will change in the near future. This is the idea of **linear approximation**.

For example, if I have 2 inches of flood water in my basement at 9:00 am and the rate at which flood water is rising at that moment is 4 inches per hour, then I expect to have 3 inches (3 inches = 2 inches + (4 inches per hour)·(1/4 hour)) of water in my basement at 9:15. However, it would be unreasonable to predict that I would have 98 inches of water (98 inches = 2 inches + (4 inches per hour)·(24 hours)) by 9:00 am the next day, because, while it is reasonable to assume that the rate at which water is rising is approximately constant over a short interval of time (like 15 minutes), it is unreasonable to expect the rate to be constant over a long interval of time (24 hours.)

This is the intuitive idea behind using a tangent line to approximate a function locally. If the height, in inches, of water in my basement t hours after 9:00 am is $h(t)$. Then the fact that there is two inches of water in my basement at 9:00 am means $h(0) = 2$, and the fact that the water is rising at a rate of 4 inches per hour at 9:00 am means that $h'(0) = 4$. Locally, the tangent line, represented by the linear function $L(t) = 2 + 4t$, is a reasonable approximation for $h(t)$. So the height of water at 9:15 is reasonably approximated as: $h(1/4) \approx L(1/4) = 2 + 4(1/4) = 3$ inches. However, the height of the water at 9:00 the following day, which is $h(24)$, cannot reasonably be approximated by $L(24) = 2 + 4(24) = 98$ inches.

Assignments

1. Reading Assignment

Read Sections 2.4 and Section 3.9, on linear approximation, focusing on the first part, "The Tangent Line Approximation," page 169. Hopefully this material is familiar to you from Calc I.

Words and phrases in *italics* are important words and phrases. Formulas in blue boxes are important formulas. Pay attention to these things and take notes on them in your notebook!

2. Discussion Problems

2.4 #1, 2, 57, 3.9 # 1, 4, 8, 17, 26

3. Practice Problems and Quality Solution

Practice: 2.4: #5, 6, 3.9 # 2, 5, 12(a)(b), 20
Quality Solution: 3.9 #6

Main Points:

1. Extending the idea of linear approximation to quadratic approximation
2. Taylor polynomials of higher degree

Overview

Recall that if a function is concave up, then a forward linear approximation will be an under-estimate. (Similarly, if a function is concave down, then a forward linear approximation will be an under-estimate.) We can improve our estimate, taking concavity into account by adding a quadratic term whose coefficient comes from the second derivative. This is the idea behind a quadratic approximation. Extending this idea, we can make better and better estimates by constructing polynomials of higher degree whose coefficients come from higher order derivatives. Such polynomials are called **Taylor polynomials**.

Assignments

1. Reading Assignment

Read Section 10.1, and take notes in your notebook. Then answer the reading questions.

2. Discussion Problems

10.1 # 3, 13, 21, 23, 25

3. Practice Problems and Quality Solution

Practice: 10.1 # 1, 15, 17, 26-29, 37
Quality Solution: 10.1 #12

Math 114, Taylor Polynomials (Section 10.1)

Name: ______________________________ **Section:** __________

Read Section 10.1, focusing on pages Taylor polynomials of degree 1 2. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Taylor Polynomials of Degree 1: Linear Approximation.** Reread Example 1.
   (a) What is the Taylor polynomial of degree 1 (i.e. the linear approximation) for $g(x) = \cos x$, with x in radians, for x near zero?

   (b) Try this: What is the Taylor polynomial of degree 1 for $h(x) = e^x$ for x near zero?

2. **Taylor Polynomials of Degree 2: Quadratic Approximation.** Reread Example 2.
   (a) What is the quadratic approximation to $g(x) = \cos x$ for x near zero?

   (b) Try this: What is the quadratic approximation to $h(x) = e^x$ for x near zero?

3. **Higher Degree Taylor Polynomials.** Reread Examples 3 and 4.
   (a) What is the Taylor polynomial of degree 8 for $g(x) = \cos x$ for x near zero?

   (b) Which Taylor polynomial, P_2 or P_8 is a better approximation for $g(x)$ near zero? Explain.

4. **Taylor Polynomials around $x = a$.** Reread Example 7.

   (a) What is the Taylor polynomial of degree 4 approximating $f(x) = \ln x$ for x near 1?

   (b) Look at the graph of $P_4(x)$ and the graph of $\ln(x)$ in Figure 10.7. On what range of x-values does it look like $P_4(x)$ approximates $\ln(x)$ well? (Eye-ball it and estimate to one decimal place.)

5. What struck you in reading this section? What is still unclear to you? What questions do you have?

Main Points:

1. The idea of a Taylor series as a Taylor polynomial of infinite degree
2. Limitations in the scope of a Taylor series approximating a function

Overview

Recall that the quadratic approximation for a function improves the linear approximation for the function by taking concavity into account, and this trend continues with higher degree Taylor polynomials: the higher the degree, the better the approximation, at least locally. In this section, we consider the family of all Taylor polynomials for a given function, centered at a given point, by looking at a **Taylor series**.

Assignments

1. Reading Assignment

Read Section 10.2, and take notes in your notebook. Then answer the reading questions.

2. Discussion Problems

10.2 # 3, 5, 18, 39(a)*
*For #39(a), you may use *Mathematica* or a graphing calculator to graph the Taylor polynomials.

3. Practice Problems and Quality Solution

Practice: 10.2 # 6, 13, 19, 29, 37*
Quality Solution: 10.2 #12
*For #37, you may use *Mathematica* or a graphing calculator to graph the Taylor polynomials.

Math 114, Taylor Series (Section 10.2)

Name: ______________________________ **Section:** __________

Read Section 10.2, focusing on "Taylor Series for $\cos x$, $\sin x$, e^x" and "Taylor Series in General." For now, you can skip over "The Binomial Series Expansion." Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Taylor Series for $\cos x$, $\sin x$, and e^x.**
   (a) What is the Taylor series for $\cos x$ centered at zero?

   (b) What is the *general term* in the Taylor expansion for $\cos x$ about $x = 0$?

2. **Taylor Series in General.** Make sure to write the formulas for Taylor series for $f(x)$ about $x = 0$ and for Taylor series for $f(x)$ about $x = a$ in your notebook.

3. **Convergence of Taylor Series.** Do not worry about the precise meaning of the word "convergence;" we will discuss this concept carefully at the end of the semester. An informal understanding of the notion is sufficient for our present purposes. Roughly speaking, if the Taylor polynomials P_0, P_1, P_2, $\ldots P_n \ldots$ give better and better approximations for a function f in a certain range of x-values, then we say that the Taylor series *converges* to f on that interval of x-values.

   Reread the passage about the Taylor series for $\ln x$ about $x = 1$ under "Convergence of Taylor Series", and look at Figure 10.10. True or false: $P_n(x)$ approximates $\ln(x)$ better and better as n gets larger, regardless of what x is. Explain.

4. What struck you in reading this section? What is still unclear to you? What questions do you have?

Main Points:

1. Estimating accumulated change over a long interval
2. Estimating the area area under a curve using left and right sums
3. The definite integral as exact accumulated change or exact area

Overview

Recall that we can approximate the change in a quantity over a short interval of time using an instantaneous rate of change. (This is linear approximation.) If we want to approximate the **accumulated change** in a quantity over a *long* interval of time, we can divide the long interval into short subintervals, estimate the changes in the quantity over each subinterval, and then add up these changes to get an estimation for the accumulated change over the long time interval. (Such a sum is called a **Riemann sum**.)

Dividing up the long interval into *more* subintervals and repeating the process *improves* our estimate.

If we have a *graph* of a positive rate function $r(t)$ versus time, then the multiplication of the rate at time $t = a$ by a short time interval Δt represents the area of a rectangle of height $r(a)$ and width Δt. Thus estimating net change can be understood as estimating the **area under a curve** using rectangles, whose heights are determined by the graph of the function. Dividing the interval into more subintervals results in more rectangles; the more rectangles we use, the better our estimate will be.

For examples with distance and velocity, see Section 5.1.

We find *exact* accumulated change and *exact* areas under curves using limits. As the number of subintervals (rectangles) increases, the approximation gets better and better; the limit is called the **definite integral**.

If a rate function $r(t)$ is *negative*, this means that the quantity is *decreasing* and the net change over a time interval is negative. Because of this, when we talk about the definite integral as the "area under the curve," we really mean that it is the **signed area between the curve and the x-axis**: the signed area is positive when the curve is above the x-axis and negative when the curve is below the x-axis.

Assignments

1. Reading Assignment.

Read 5.1 and 5.2. Take notes, and answer the reading questions.

2. Discussion Problems.

5.1 # 17, 18, 25, 47, 48, 5.2 # 34, 58

3. Practice Problems and Quality Solution.

Practice: 5.1 # 12, 19, 20 31, 33, 5.2 # 8
QS: 5.1 #24

Math 114, The Definite Integral: Area and Net Change (Sections 5.1, 5.2)

Name: ______________________________ **Section:** __________

Read Sections 5.1 and 5.2. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Distance and Velocity.** Review the notation in "Left and Right Sums", and try 5.1 #1, 23.

2. **The Definite Integral.** Try 5.2 #1, 2, 6.

3. What struck you in reading these sections? What is still unclear? What questions do you have?

Main Points:

1. Midpoint Rule, Trapezoidal Rule
2. Over/under estimates, errors
3. Simpson's Rule

Overview

Recall that we can estimate the value of a definite integral using a left sum, which uses rectangles whose heights are determined by the value of the function on the *left* endpoints of the subintervals, or a right sum, which uses rectangles whose heights are determined by the value of the function the *right* endpoints.

In this section, we refine our estimates using more sophisticated numerical methods. The **midpoint rule** uses rectangles whose heights are determined by the value of the function at the *midpoints* of the subintervals. The **trapezoidal rule** takes the *average* of the left and right sums. **Simpson's rule** takes a *weighted average* of the estimates obtained from the midpoint and trapezoidal rules.

The reasons for averaging the left and right sums in the trapezoidal rule and for taking a weighted average of the midpoint and trapezoidal rule in Simpson's rule become apparent when we look at the **errors** of our estimates. For example, for an increasing function the left sum *underestimates* the integral (a positive error) whereas the right sum *overestimates* the integral (a negative error). This is the motivation for *averaging* the two (to cancel out the errors) in the trapezoidal rule.

Assignments

1. Reading Assignment

Read Section 7.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.5 # 1, 20, 21, and one additional problem; see below.

3. Practice Problems and Quality Solution

Practice: 7.5 # 4, 22, and one additional problem; see below.
Quality Solution: 7.5 #24

Math 114, Numerical Methods for Definite Integrals (Section 7.5)

Name: ______________________________ **Section:** __________

Read Section 7.5, focusing on the midpoint rule and the trapezoidal rule. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Midpoint and Trapezoid Rules.** Reread Examples 1 and 2.

   (a) For $\int_1^2 \frac{1}{x}\,dx$, list the vlaues of LEFT(2), RIGHT(2), MID(2), TRAP(2) in ascending order (from smallest to largest.)

   (b) Which of the four estimates are underestimates? Which are overestimates? Which estimate is the best?

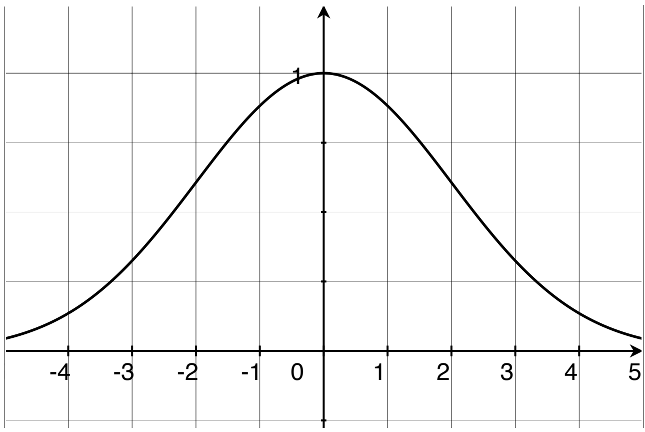
2. **Over- and Underestimates.** Reread pages 378-379.

   (a) If $f(x)$ is an *increasing* function, which of LEFT(n), RIGHT(n), MID(n), TRAP(n) for $\int_a^b f(x)dx$ is guaranteed to be an underestimate? an overestimate?

   (b) If $f(x)$ is *concave up*, which of LEFT(n), RIGHT(n), MID(n), TRAP(n) for $\int_a^b f(x)dx$ is guaranteed to be an underestimate? an overestimate?

3. The graph of $e^{-x^2/8}$ for $-5 \le x \le 5$ is shown below. Shade the region whose area is given by the definite integral $\int_0^2 e^{-x^2/8}\, dx$.

   (a) Given the shape of the graph, which of the four rules should give overestimates of the integral and which should give underestimates?



   (b) Compute LEFT(2), RIGHT(2), MID(2), and TRAP(2). (Round to six decimal places.)

4. What struck you in reading these sections? What is still unclear? What questions do you have?

Additional Discussion Problem

We will approximate the value of $\int_0^1 \cos(x^2)\,dx$. Use *Mathematica* for your computations.

(a) Use *Mathematica* to graph the function: `Plot[Cos[x^2], {x, 0, 1}]`. Sketch the graph below. In particular, make sure you have the concavity correct.

(b) Approximate the integral using LEFT(4), RIGHT(4), MID(4), and TRAP(4). Use *Mathematica*, and round to six decimal places.

(c) Use your sketch of the graph in (a) to determine which of your estimates in (b) are overestimates and which are underestimates. Which is your best underestimate? Which is your best overestimate?

(d) Approximate the integral using SIMP(4).

Additional Practice Problem

In this problem we will estimate $I = \int_0^1 \sin(\frac{1}{2}x^2)\,dx$.

(a) Use *Mathematica* to graph the function: `Plot[Sin[(1/2)x^2], {x, 0, 1}]`. Sketch the graph below. In particular, make sure you have the concavity correct.

(b) List the values LEFT(n), RIGHT(n), MID(n), and TRAP(n), and I in increasing order (smallest to largest).

(c) Use *Mathematica* to compute LEFT(5), RIGHT(5), MID(5), TRAP(5), and SIMP(5).

Main Points:

1. FTC 1: Evaluating definite integrals using antiderivatives.
2. FTC 2: Constructing antiderivatives using the definite integral.

Overview

Recall that, given a function $r(t)$ for the rate at which some quantity Q is changing, we can *estimate* the net change in Q over a long time interval $[a, b]$ by dividing up the long interval into short subintervals, using linear approximation to estimate the change in Q over each subinterval, and adding up these changes. The *exact* accumulated net change is obtained by taking a limit of Riemann sums; this limit is the definite integral: $\Delta Q = Q(b) - Q(a) = \int_a^b r(t)\, dt$.

This is the idea behind the **first part of the Fundamental Theorem of Calculus**, which says that if $F(x)$ is a function with a continuous derivative $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$. Since f is a derivative of F, we say F is an antiderivative of f. This means that we can evaluate definite integrals *exactly* whenever we can find *antiderivatives*.

This same insight allows us to *construct* antiderivatives. Again, suppose $r(t)$ is a function that tells the rate at which some quantity Q is changing. Now suppose we want to know the accumulated net change in Q for many different time intervals. We fix a given intial time $t = a$ and let $A(x)$ be the net change from $t = a$ to $t = x$, for many different x-values. Graphically, this is represented by finding the (signed) area between the graph of r and the t-axis from a to x. So $A(x)$ is sometimes called the "area-so-far" function. We know that this area is represented by a definite integral: $A(x) = \int_a^x r(t)\, dt$.

Since $A(x)$ tells the net change in Q from $t = a$ to $t = x$, We can find a formula for $Q(x)$ if we know the initial quantity Q_0:

$$Q(x) \;=\; Q_0 \;+\; \int_a^x r(t)\, dt \;=\; Q_0 + A(x)$$

Since r is the derivative of Q, Q is an antiderivative of r. Notice that every choice of initial value Q_0 will give an antiderivative for r. In particular, choosing $Q_0 = 0$ shows that the area-so-far function itself is an antiderivative for r.

This is the idea behind the **second part of the Fundamental Theorem of Calculus**, which says that, for a continuous function $f(x)$, we can construct an antiderivative function $F(x)$ as follows: choose a number a in the domain of f and let $F(x) = \int_a^x f(t)\, dt$.

Assignments

1. Reading Assignment

Read Sections 5.3, 6.1, and 6.4. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

5.3 # 16, 21, 6.1 # 8, 6.4 # 3*, 9
*For 6.4 #3, replace the lower limit with 0.00001, as in Example 1, page 342, and use SIMP(2).

3. Practice Problems and Quality Solution

Practice: 5.3 # 23, 6.1 # 3, 11, 6.4 # 8
Quality Solution: 6.4 #22

Math 114, The Fundamental Theorem of Calculus (Sections 5.3, 6.1, and 6.4)

Name: ______________________________ **Section:** __________

Read Sections 5.3, 6.1, and 6.4. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Applications of the Fundamental Theorem.** Reread Example 3 in Section 5.3.

   (a) Suppose that one of your classmates says that neither car is ahead after one minute, because their graphs intersect at $t = 1$. Explain, in your own words, why this is incorrect.

   (b) Try Exercise 8, at the end of Section 5.3.

2. **Computing Values of an Antiderivative Using Definite Integrals** Reread this part of Section 6.1, focusing on Example 4 and Example 5. Try 6.1 #1 and 2.

3. **Constructing Antiderivative Functions** Reread Section 6.1, focusing on the initial discussion up to and including Theorem 6.2.

   (a) Use the Construction Theorem for Antiderivatives to write down a formula for an antiderivative $F(x)$ of $f(x) = e^{-x^2}$. (Your formula should have an integral in it.)

   (b) This antiderivative function is important in the theory of probability. Use LEFT(2) or RIGHT(2) to estimate $F(1)$. Round your answer to six decimal places.

4. What struck you in reading these sections? What is still unclear? What questions do you have?

Main Points:

1. New terms: differential equation, general solution of differential equation
2. Finding solutions to simple differential equations, verifying solutions to differential equations
3. New terms: initial value problem, initial condition, particular solution, equillibrium solution
4. Solving an initial value problem, given the general solution and an initial condition

Overview

A **differential equation** is simply an equation involving a derivative. A simple example is the equation $\frac{dy}{dx} = 2x$. Notice that this equation is true for the function $y = x^2$ and the function $y = x^2 + 5$, since these are antiderivatives of $2x$. Such functions are called **particular solutions** of the differential equation. The most general form of the solution is, of course, $y = x^2 + C$, where C is a real number. This is called the **general solution**. It represents a whole *family* of solutions. Section 6.3 discusses differential equations like this, that can be solved using antiderivatives.

Another simple differential equation is $\frac{dy}{dx} = 2y$. This equation *cannot* be solved using antiderivatives as above, but we might be able to solve it by gussing and checking. Can you think of a function whose derivative is simply 2 times itself? How about $y = e^{2x}$? We can check that this is a solution simply by differentiating: $\frac{d}{dx}\, e^{2x} = 2 \cdot e^{2x}$. This shows that, when $y = e^{2x}$, $\frac{dy}{dx} = 2y$, i.e. $y = e^{2x}$ is a solution to the differential equation. It turns out that every function of the form $y = Ce^{2x}$, where C is a real number, is also a solution, as we can check: $\frac{d}{dx}\, Ce^{2x} = C(2e^{2x}) = 2 \cdot (e^{2x})$.

We can find particular solutions from the general solution, if we also have an **initial condition**. For example, if the general solution is $y = Ce^{2x}$ and we have the initial condition $y(0) = 5$, we can find the particular solution by substituting in $x = 0$ and $y = 5$ and solving for C as follows: $5 = Ce^{2\cdot 0} = Ce^0 = C$. Thus the particular solution is $y = 5e^{2x}$. A problem of this sort is called an **initial value problem**.

Even without a formula for the general solution of a differential equation, we can often determine quite a bit about the family of solutions: we can use qualitative analysis to sketch graphs and numerical methods to generate tables of data approximating the solution. Sections 11.1-11.3 discuss these methods in depth.

Assignments

1. Reading Assignment

Read Sections 6.3 and 11.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

6.3 # 5, 7, 12,   11.1 # 9, 11, 18, 20

3. Practice Problems and Quality Solution

Practice: 6.3 # 8, 10, 16*, 11.1 # 8, 17, 21
*For 6.3.16, your final answer will be in terms of k.
Quality Solution: 11.1 #10

Math 114, Introduction to Differential Equations (Sections 6.3, and 11.1)

Name: ______________________________ **Section:** __________

Read Section 6.3, focusing on the first part (before the part on equations of motion), and Section 11.1, focusing on the first part (before the part on first and second-order differential equations). Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Solving Simple Differential Equations with Antiderivatives.** Reread Examples 1 and 2 in Section 6.3. Imitate these examples to solve the following problems.

   (a) Find the general solution of the differential equation $\frac{dy}{dx} = 3x^2$.

   (b) Find the solution of the initial value problem $\frac{dy}{dx} = 3x^2$; $y(1) = 0$.

2. **How Fast Does a Person Learn?** Reread the first part of Section 11.1, "How Fast Does a Person Learn?", and answer the following questions.

   (a) Here, y stands for the percentage of the task already mastered. What does $\frac{dy}{dt}$ mean, in terms of learning the task? What happens to $\frac{dy}{dt}$ as y increases? Can you see this behavior in the graphs in Figure 11.1?

   (b) **Solving the Differential Equation Numerically** Suppose that, at the beginning of the employee's training, the employee has mastered 0% of the task. Use the numerical method that generates the data in Table 11.1 to find the y-value for $t = 6$.

(c) **A Formula for the Solution to the Differential Equation** According to the text, what is the general solution to the differential equation?

(d) **Finding the Aribitrary Constant** If we suppose that, at the beginning of the employee's training, the employee has mastered 0% of the task, we have the initial conditions: $y = 0$ when $t = 0$. What is the particular solution of the differential equation in this case?

Find the three particular solutions corresponding to the initial values: $y(0) = 50$, $y(0) = 100$, and $y(0) = 150$.

Which of the above solutions have meaningful practical interpretations, in terms of learning?

3. What struck you in reading these sections? What is still unclear? What questions do you have?

Main Points:

1. Generating a slope field from a differential equation
2. Sketching solution curves using a slope field

Overview

Given a differential equation that expresses $\frac{dy}{dx}$ in terms of x and y, we can generate a table of values with x and y as inputs and $\frac{dy}{dx}$ as outputs. Thus, for each point (x, y) in the plane, we can sketch the slope, $\frac{dy}{dx}$, of the solution curve passing through that point (as long as it is defined). The collection of all these short lines is called a **slope field**, and it gives use a way to sketch solution curves for the differential equation.

Assignments

1. Reading Assignment

Read Section 11.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

11.2 # 6, 11, 19, 23

3. Practice Problems and Quality Solution

Practice: 11.2 # 5, 20, 21
Quality Solution: 11.2 # 12

Name: ______________________________ **Section:** __________

Read Section 11.2, focusing on the first part (before the part about existence and uniqueness of solutions). Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. Consider the differential equation $\frac{dy}{dx} = x - y + 1$.

   (a) Complete the table below by choosing x and y-values and finding $\frac{dy}{dx}$ at those values using the differential equation. For, example, with $x = 0$ and $y = 0$, we have $\frac{dy}{dx} = 0 - 0 + 1 = 1$.

   (b) Sketch a slope field for this differential equation, using your table of values by drawing a short line with the appropriate slope $\frac{dy}{dx}$ at that point. For example, at the point $(0, 0)$, draw a short line with slope 1, since $\frac{dy}{dx} = 1$ when $x = 0$ and $y = 0$.

   (c) Use your slope field to sketch three significantly different solution curves.

x	y	$\frac{dy}{dx}$
0	0	1
0	1	0
0	2	-1
0	3	
0	4	
0	-1	
0	-2	
0	-3	
-1	0	
-1	1	

2. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Using separation of variables to find the general solution of certain differential equations
2. Writing a differential equation to model a real-life situation
3. Stable and unstable equillibrium solutions

Overview

A separable differential equation is one that can be written in the form: $y' = f(x)g(y)$. For example:

$$y' = xy^3 \qquad y' = x^2\,y^{-2} \qquad y' = 6x^2/(2y + \cos y)$$

are separable differential equations. To solve a separable differential equation, write the derivative in Leibniz notation (dy/dx instead of y'), write the differential equation in "differential form," i.e. with all the x-values on one side, with dx and all the y-values on the other side, with dy, and integrate both sides.

An example of a separable differential equation that occurs in application is the equation modelling unconstrained population growth. This model operates under the assumption that a population will grow at a rate directly proportional to the size of the population, P. In other words $\frac{dP}{dt}$ is proportional to P.

$$\frac{dP}{dt} = k\,P \qquad \text{(for some } k > 0)$$

The positive constant k is the constant relative growth rate of the population.

According to Newton's Law of Cooling, the rate at which temperature of an object decreases is proportional to the temperature difference between the object and its surroundings. Thus the temperature of a cooling object can also be modelled by a differential equation involving a proportionality statement.

In general, when a quantity A is *directly proportional* to a quantity B, that means that there is a positive constant, say k, such that $A = kB$. The constant k is called the *constant of proportionality.*

An equillibrium solution of a differential equation is a constant solution. Equillibrium solutions for a differential equation of the form $y' = g(y)$ can be found by letting $y' = 0$ and solving for y, since this enables us to find y-values at which the rate at which y is changing is zero.

Assignments

1. Reading Assignment

Read Sections 11.4 and 11.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

11.4 # 7, 15, 18, 11.5 # 8, 43

3. Practice Problems and Quality Solution

Practice: 11.4 # 10, 11.5 # 28, 41
Quality Solutions: 11.4 # 20, 11.5 # 40*
*11.5.40 is a Challenge Problem. It will be graded on a more lenient scale.

Math 114, Separation of Variables and Modeling (Sections 11.4 and 11.5)

Name: ____ **Section:** __________

Read Section 11.4, focusing on the first part (before the part about the justification for separation of variables), and Section 11.5, focusing on the population growth example and the heating/cooling examples. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Separation of Variables.** Reread Examples 1 and 2 in Section 11.4.

   (a) What must be done to the differential equation before the integration step?

   (b) What must be done after the integration step to find the general solution?

   (c) Imitate Example 1 to solve the initial value problem: $\frac{dP}{dt} = .02P\,;\ P(0) = 100.$

   (d) Imitate Example 2 to solve the initial value problem: $\frac{dH}{dt} = -0.3(H - 20)\,;\ H(0) = 100.$

2. **Setting up a Differential Equation for Population Growth** Reread the beginning of Section 11.5, about setting up a differential equation for unrestricted population growth.

   (a) In this model, the rate at which the population grows with respect to time is proportional to what quantity?

   (b) What does $\frac{dP}{dt}$ represent in the differential equation? What does 0.02 represent? P?

3. **A Differential Equation for Heating and Cooling** In Section 11.5, reread the first two paragraphs in the part entitled "Newton's Law of Heating and Cooling" and the part entitled "Equilibrium Solutions."

   (a) In this model, the rate at which the coffee cools with respect to time is proportional to what quantity?

   (b) Suppose the coffee stands in a room where the temperature is 20°C. If $H(t)$ is the temperature, in °C, of the coffee at time t, in minutes, the differential equation modeling the cooling of the coffee is the differential equation

   $$\frac{dH}{dt} = -k(H - 20)$$

   In this equation what does $\frac{dH}{dt}$ represent? What does $(H - 20)$ represent?

   What is the equilibrium solution? Is it stable or unstable?

4. What struck you in reading these sections? What is still unclear? What questions do you have?

Main Points:

1. Guess-and-check to "undo the chain rule"
2. Changing variables in an integral: choose inside function "w" and then $dw = w'(x)dx$
3. Two methods for using substitution in definite integrals

Overview

So far the only strategy we have for finding antiderivatives is to recognize them as derivatives of familiar functions, sometimes using algebra or trigonometry to rewrite a function first. Can you recognize $2e^{2x}$ as the derivative of a familiar function? It is the derivative of e^{2x}. The constant factor of 2 comes from the chain rule. For very simple examples, we can "undo the chain rule" in this way. (See Examples 1 and 2.)

For less simple examples, it helps to perform a change of variables. We give the "inside function" a name, say $w(x)$ and transform an integral having x as the variable of integration to an integral having w as the variable of integration. Remember that the chain rule says

$$\frac{d}{dx} F(w(x)) \quad = \quad F'(w(x)) \cdot w'(x)$$

Thus, if F is an antiderivative for f (i.e. $F' = f$),

$$\int f(w)\, w'(x)\, dx \;=\; \int f(w)\, dw \;=\; F(w(x)) \;+\; C$$

since $w'(x)\, dx = dw$. (See Examples 3-7.)

We can sometimes use substitution even if the integrand is not a constant multiple of something of the form $f(w(x))\, w'(x)$. In particular, as long as the integrand can be rewritten as $w'(x)$ times something entirely in terms of w, substitution is worth trying. See Examples 12-13.

Since substitution is a technique for finding antiderivatives, it is also useful for definite integrals. The trick is to be careful not to plug in x-values for w. There are two ways to do this. A two-step method requires finding the indefinite integral first; as an alternative, you can transform the limits of integration along with the whole integral. See Examples 9-11.

Assignments

1. Reading Assignment

Read Section 7.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.1 # 19, 27, 36, 40, 59, 63, 77

3. Practice Problems and Quality Solution

Practice: 7.1 # 10, 22, 31, 33, 60, 74, 79, 148
Quality Solution: 7.1 # 66

Math 114, Integration by Substitution (Section 7.1)

Name: ____________________________________ **Section:** ________

Read Section 7.1. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Simple Substitutions.** Reread Examples 1-4.

   (a) Imitate Example 1 (or Example 3) to find $\int 2xe^{x^2}\,dx$.

   (b) Imitate Example 2 (or Example 4) to find $\int x^4 \cos(x^5)\,dx$.

2. **Less Simple Substitutions.** Reread Examples 5-7.

   (a) Find $\int \cos^5 x \sin x\,dx$. (Hint: Let $w = \cos x$.)

(b) Find $\int e^x \sqrt{1+e^x}\, dx$. (Hint: Let $w = 1 + e^x$.)

3. **Substitution with a Definite Integral** Reread Examples 9-11.

   (a) State the two ways to use substitution with a definite integral. (See the box after Example 9.)

   (b) Find $\int_1^2 2xe^{x^2}\, dx$ using *both* ways.

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. IBP as "reversing the product rule" to exchange a hard integral for an easier one
2. Two tricks: letting $v' = 1$; noticing a pattern of repeating derivatives

Overview

Integration by parts is a way to use the "reverse product rule" to exchange a hard integral for an easier one. Recall that the product rule can be written as:

$$\frac{d}{dx}\, u(x)\, v(x) \quad = \quad u'(x)\, v(x) \ + \ u(x)\, v'(x)$$

Restating in terms of integrals and rearranging gives:

$$\int u(x)\, v'(x)\, dx \quad = \quad u(x)\, v(x) \ - \ \int u'(x) v(x)\, dx$$

Using the shorthand $du = u'(x)\, dx$ and $dv = v'(x)\, dx$, we can rewrite this as:

$$\boxed{\int u\, dv \ = \ uv \ - \ \int v\, du}$$

IBP is a good strategy to try when the integrand is a product of two functions. In order for IBP to work, you need to be able to differentiate one of the functions and anti-differentiate the other. Choose u to be the function you want to differentiate and v' to be the function you want to anti-differentiate.

Sometimes IBP can be used even when the integrand does not look like a product of two functions. In particular, if we know the derivative of the integrand, we can let the whole integrand be u and we can let $v' = 1$. For example, this works for $\int \ln x\, dx$ and $\int \arcsin x\, dx$. See Example 3.

Sometimes IBP can be used even when neither part of the integrand becomes simpler when differentiated, if we can notice a pattern of repeating derivatives. See Examples 6-7.

Assignments

1. Reading Assignment

Read Section 7.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.2 # 5, 6, 9, 13*, 58, 2, 15*, 60*
*Hints: For #13, imitate Example 6. For #15, let $u = (\ln t)^2$ and $dv = dt$; also see Example 3. For #60, imitate Example 7.

3. Practice Problems and Quality Solution

Practice: 7.2 # 8, 10, 20, 26, 59, 73, 61
Quality Solution: 7.2 # 16

Math 114, Integration by Parts (Section 7.2)

Name: ____________________________________ **Section:** ________

Read Section 7.2, focusing on Examples 1-5. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Straightforward IBP.** Reread Examples 1, 2, 4, and 5.

   (a) Imitate Examples 1 and 2 to find $\int x \sin x \, dx$.

   (b) Imitate Example 4 to find $\int x^2 \ln(x) \, dx$.

   (c) Imitate Example 5 to find $\int x^2 \sin(x) \, dx$. (Use IBP twice.)

2. **An antiderivative for** $\ln x$**.** Reread Example 3.

   (a) What are u and v' in this example?

   (b) What is the antiderivative of $\ln x$?

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Finding simple partial fractions decompositions by hand
2. Using partial fractions decompositions to simplify integration

Overview

The method of **partial fractions** is an algebraic technique that can be helpful for integration. In particular, the partial fractions decomposition is a way to *rewrite* a rational function as a *sum* of *simpler* rational functions, as long as the degree of the numerator is smaller than the degree of the denominator. (If the degree of the denominator is larger than the degree of the numerator, long division of polynomials can be used first. See Example 5.) It is a reverse process to adding rational functions, and as such requires "undoing the common denominator."

We use the partial fractions decomposition to rewrite rational integrands as sums of simpler rational functions. To evaluate these simpler integrals it may be necessary to use a substitution. Recall some basic antiderivatives:

$$\int \frac{dx}{x} = \ln|x| + C\,; \qquad \int \frac{dx}{x^p} = \frac{1}{(1-p)\,x^{p-1}} + C\,, \quad (p>1)\,; \qquad \int \frac{dx}{1+x^2} = \arctan(x) + C$$

Assignments

1. Reading Assignment

Read Section 7.4, up to but not including the part about Trigonometric Substitutions. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.4 # 15, 17, 38, 47

3. Practice Problems and Quality Solution

Practice: 7.4 # 16, 40, 48, 73
Quality Solution: 7.4 # 46

Math 114, Method of Partial Fractions (Section 7.4-I)

Name: ______________________________ **Section:** __________

Read Section 7.4, up to but not including the part about Trigonometric Substitutions, focusing on Examples 1-4. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Denominator is Product of Distinct Linear Factors.** Reread Examples 1 and 2. We will imitate these examples to evaluate $\int \frac{2x+3}{x^2-4x-5}\,dx$.

   The denomniator factors as $(x+1)(x-5)$, and the partial fractions decomposition is of the form

   $$\frac{2x+3}{x^2-4x-5} \quad = \quad \frac{A}{x+1} + \frac{B}{x-5}$$

   (a) Clear denominators by multiplying both sides by $(x+1)(x-5)$.

   (b) Expand/foil the right hand side, and collect terms.

   You should get: $(A+B)x \ + \ (B-5A)$. This means that

   $$2x+3 \quad = \quad (A+B)x \ + \ (B-5A)$$

   Equating constant terms and coefficients of x gives: $2 = A+B$ and $3 = B-5A$. Use these two equations to solve for A and B.

   (c) Use the partial fractions decomposition to evaluate $\int \frac{2x+3}{x^2-4x-5}\,dx$.

2. **Denominator has a Repeated Linear Factor.** Reread Example 3. We will imitate this example to evaluate $\int \frac{x-9}{(x+5)(x-2)^2}\, dx$.

   Since there is a repeated linear factor, the partial fractions decomposition is of the form:

$$\frac{x-9}{(x+5)(x-2)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

   If we multiply both sides by $(x+5)(x-2)^2$ and then equate constant terms and coefficients of x, we get $A = -2/7$, $B = 2/7$, $C = -1$. Use this partial fractions decomposition to evaluate the integral.

3. **Denominator has a Quadratic that Cannot be Factored.** Reread Example 4. Suppose we wished to find the partial fractions decomposition for $\frac{x-9}{(x+5)(x^2-3x+9)}$.

   (a) Use the quadratic formula to show that $x^2 - 3x + 9$ cannot be factored.

   (b) What is the correct form of the partial fractions decomposition in this case? (You do not need to find the coefficients A, B, and C.)

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Basic trig substitution: when integrand is similar to the derivatives $\arcsin x$ or $\arctan x$.
2. More general trig substitution: when integrand involves $x^2 + a^2$ or $\sqrt{a^2 - x^2}$; using triangle.
3. Completing the square before using a trig substitution

Overview

Recall that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

For integrals that are very similar to these, a simple w-substitution or a basic trig substitution can be used, as in Examples 7 and 10:

Ex. 7: $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin(\tfrac{x}{2}) + C$ **Ex. 10:** $\displaystyle\int \frac{1}{9+x^2}\,dx = \tfrac{1}{3}\arctan(\tfrac{x}{3}) + C$

More generally, when an integral involves something of the form $\sqrt{a^2 - x^2}$ or $a^2 + x^2$, a trig substitution may be useful.

A trig substitution looks a little different from the simple w-substitutions we discussed in 7.1; instead of identifying something in the integrand as an "inside function," we let x be a trig function in terms of θ then change the variable of integration from x to θ. We may use a **trig identity** or an appropriate **right triangle** (using SOH-CAH-TOA and the Pythagorean Theorem) to simplify the integrand before integrating. Often our new integrand involves integer powers of $\sin\theta$ and/or $\cos\theta$. The formulas in Part IV of the table of integrals at the end of the book may be useful.

Finally, trig substitutions can also be useful after **completing the square** to rewrite part of the integrand in the form $\sqrt{a^2 - (x-h)^2}$ or $a^2 + (x-h)^2$, as in Examples 12 and 13.

Assignments

1. Reading Assignment

Read the part about Trigonometric Substitutions in Section 7.4. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.4 # 20, 21, 23, 54, 59*
*Use one of the formulas from IV in the table of integrals at the back of the book.

3. Practice Problems and Quality Solution

Practice: 7.4 #22, 24, 30, 55*, 72*
Quality Solution: 7.4 # 60*
*Use one of the formulas from IV in the table of integrals at the back of the book.

Math 114, Trigonometric Substitutions (Section 7.4-II)

Name: ______________________________ **Section:** __________

Read the part of Section 7.4 about Trigonometric Substitution. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. In each of the examples, a trig identity ("the Pythagorean identity") is used to simplify the integrand after making a substitution. What is this identity? (See the paragraph before Example 6.)

2. For each of the examples listed below, state the trig substitution that is used. In particular, state both x and dx.

   (a) Example 6:

   (b) Example 7:

   (c) Example 8:

   (d) Example 10:

   (e) Example 11:

   (f) Example 12:

3. **Using a Triangle.** Reread Examples 8 and 9.

   (a) Draw and label the triangle used in Example 9. (Make sure you label the angle θ.)

   (b) What is $\cos\theta$, in terms of x?

   (c) What would $\tan\theta$ be, in terms of x, if we needed to find it?

4. For each expression below, state an appropriate trig substitution that could be used to simplify. State both x and dx.

   (a) $\sqrt{a^2 - x^2}$

   (b) $a^2 + x^2$

   (c) $\sqrt{a^2 - (x-h)^2}$

   (d) $a^2 + (x-h)^2$

5. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Limit as the technical underpinning of calculus
2. Limits describing local behavior of functions
3. Limits describing end behavior of functions

Overview

The formal mathematical notion of a **limit** is the essential technical idea underlying calculus: it is necessary for careful discussions of continuous functions, derivatives, and definite integrals, which we have studied, as well as for the convergence of improper integrals, infinite sequences, and infinite series, which we have yet to study.

Limits are useful for describing **local behavior** of functions, especially where they are undefined. For example, if we notice that the y-values of $\frac{\sin x}{x}$ seem to get closer to $y = 1$ as the x-values get closer and closer to zero, we say that the limit of $\frac{\sin x}{x}$ as x approaches zero is 1. Note that this describes the behavior of the function *near but not at* the number $x = 0$. In this example, the function is undefined at $x = 0$, but this does not negate the observable trend that the y-values are approaching 1; it indicates that the graph of has a *hole* at $x = 0$. (See Example 1.)

Definition Suppose $f(x)$ is defined on some interval around c, except perhaps at the point $x = c$. Then we write $\lim_{x\to c} f(x) = L$ and say *the limit of $f(x)$, as x approaches c, equals L* if we can make the values of $f(x)$ as close to L as we like by taking x sufficiently close to c (on either side of c) but not equal to c.

Similarly we can talk about the limit from the left and the limit from the right, if we only mean to discuss x approaching c from the left, or right, respectively.

When a function increases without bound (informally: the values "go to infinity") we use the infinity symbol (∞) to denote the limit, even though the limit technically does not exist, because the values of the function do not approach a specific number.

Limits are also useful for describing the **end behavior** of functions, i.e. what happens as x becomes larger and larger (positive or negative). If a function f approaches a specific number L as x gets larger and larger (positive), we say that *the limit of $f(x)$ as x approaches infinity is L* and we write $\lim_{x\to\infty} f(x) = L$. Similarly, we write $\lim_{x\to-\infty} f(x) = L$ if f approaches L as x becomes larger and larger negative.

Assignments

1. Reading Assignment

Read Sections 1.7, 1.8, 1.9. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

1.7 # 3; 1.8 # 3, 5; 1.9 # 4, 6, 27, 29, 63, 74, 79, 80

3. Practice Problems and Quality Solution

Practice: 1.8 # 4, 16-21; 1.9 # 5, 7, 28, 65, 75, 81, 83
Quality Solution: 1.8 #54

Math 114, Limits (Sections 1.7, 1.8, 1.9)

Name: ______________________ **Section:** ________

Read Sections 1.7, 1.8, and 1.9 focusing on the following parts: “The Idea of a Limit,” “Definition of Limit,” and “When Limits Do Not Exist” in Section 1.7, “One-Sided Limits” and “Limits and Asymptotes” in Section 1.8, “Limits and Quotients,” “Limits of the Form 0/0 and Holes in Graphs,” and “Calculating Limits at Infinity” in Section 1.9. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Local Behavior** For each item below, sketch a graph of a function with the desired properties. (Examples in the section will work for some of these, but you could also create your own examples.)

   (a) $f(0)$ is undefined; $\lim_{x\to 0} f(x) = 1$

   (b) $f(0)$ is undefined; $\lim_{x\to 0^-} f(x) = -\infty$; $\lim_{x\to 0^+} f(x) = \infty$

   (c) $f(0)$ is undefined; $\lim_{x\to 0} f(x)$ does not exist

   (d) $f(0)$ is undefined; $\lim_{x\to 0^-} f(x) = -1$; $\lim_{x\to 0^+} f(x) = 1$

2. **End Behavior.** Use what you know about the graphs of the functions to investigate the limits at infinity.

   (a) $\lim\limits_{x\to\infty} e^x$, $\lim\limits_{x\to-\infty} e^x$

   (b) $\lim\limits_{x\to\infty} x^3$, $\lim\limits_{x\to-\infty} x^3$

   (c) $\lim\limits_{x\to\infty} \ln|x|$, $\lim\limits_{x\to-\infty} \ln|x|$

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Using l'Hopital's rule to evaluate limits of quotients
2. Using limits of quotients to describe dominance
3. Variations on l'Hopital's Rule

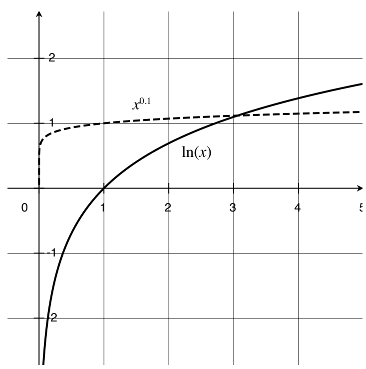
Overview

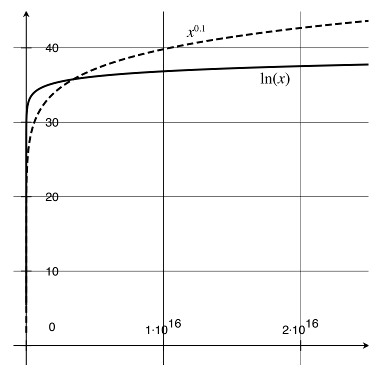
This section gives us a way to evaluate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The trick is to use **l'Hopital's rule**, which says that you can take the derivative of the top and the derivative of the bottom and *then* take the limit of *that.* Intuitively, if both the numerator and the denominator are shrinking (or growing), we use derivatives to tell us which one is shrinking (or growing) at a *faster rate.*

One application is determining the **dominance** of one function over another. For example, we can use l'Hopital's rule to prove that although $\ln x$ and x^p (for $p > 0$) both grow without bound as x increases, the power function will always surpass the logarithmic function, eventually. We do this by looking at their quotient and taking a limit as x approaches infinity.

$$\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{p \cdot x^{p-1}} = \lim_{x\to\infty} \frac{1}{p \cdot x^p} = 0$$

Note that the limit is of the form $\frac{\infty}{\infty}$, so it is valid to apply l'Hospital's rule. The $\ln x$ in the numerator is trying to make the limit go to infinity, but the x^p in the denominator is trying to make the limit go to zero; it is a competition, and l'Hopital's rule tells us who wins: since the final limit is zero the power function in the denominator wins. This means the power function dominates the logarithmic function. This is somewhat surprizing since for small p-values the power function does not appear to grow very quickly.





It is sometimes possible to use l'Hopital's Rule to evaluate limits of the form $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, or 1^∞, but it is necessary to rewrite the function as a *quotient* so that the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying l'Hopital's rule. See Examples 6, 7, and 8.

Assignments

1. Reading Assignment

Read Section 4.7. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.7 # 13, 17, 18, 41, 61, 71, 72

3. Practice Problems and Quality Solution

Practice: 4.7 # 16, 35, 42, 73, 74
Quality Solution: 4.7 # 36

Math 114, L'Hopital's Rule (Section 4.7)

Name: ______________________________ **Section:** __________

Read Section 4.7, focusing on pages 242-245. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **l'Hopital's Rule for Quotients.** Reread Examples 1, 2, and 3.

   (a) Imitate Example 1 to evaluate $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$.

   (b) Imitate Example 2 to evaluate $\lim_{x \to \infty} \frac{e^x - e^{-x}}{x}$.

2. **Dominance.** Consider the functions $f(x) = e^x - e^{-x}$ and $g(x) = x$.

   (a) Evaluate $\lim_{x \to \infty} \frac{g(x)}{f(x)}$.

   (b) Given your answer to 2(a) and 1(b), which function dominates the other: $f(x)$ or $g(x)$?

3. **Rewriting Indeterminate Forms.** Reread Example 6.

   (a) What is the form of the limit in this example?

   (b) How do we rewrite the limit so as to be able to use l'Hopital's rule?

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Two kinds of improper integrals
2. Using limits to describe improper integrals

Overview

Our original discussion of the definite integral does not allow for integrating over an infinite interval or integrating functions that are unbounded at a point, but such integrals, called **improper integrals** do arise in applications. We use limits to describe such integrals: some of which have a finite value, some of which do not. An improper integral with a finite value is called **convergent**, whereas an improper integral without a finite value is called **divergent**.

Assignments

1. Reading Assignment

Read Section 7.6. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.6 # 3, 5, 7, 27, 28, 48

3. Practice Problems and Quality Solution

Practice: 7.6 # 4, 6, 9, 13, 21, 22
Quality Solution: 7.6 # 42

Math 114, Improper Integrals (Section 7.6)

Name: ____ **Section:** __________

Read Section 7.6. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Improper Integrals.** Reread the first paragraph of the section.

   (a) What two things did we assume in our original discussion of the definite integral $\int_a^b f(x)dx$?

   (b) What are the two kinds of improper integrals?

2. **When the Limit of Integration is Infinite.** Reread the discussion of the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ at the beginning of the section as well as Example 1, about $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$, and the paragraph following it.

   (a) Which of these improper integrals converges? Which diverges?

   (b) What is the difference between the functions $1/x^2$ and $1/\sqrt{x}$ that makes the area under the graph of $1/x^2$ aproach 1 as $x \to \infty$, whereas the area under $1/\sqrt{x}$ grows very large?

3. Reread Example 3, about the family of improper integrals $\int_1^\infty \frac{1}{x^p}\,dx$.

   (a) Does $\int_1^\infty \frac{1}{x}\,dx$ converge? If so, what is its value?

   (b) Does $\int_1^\infty \frac{1}{x^3}\,dx$ converge? If so, what is its value?

4. **When the Integrand Becomes Infinite.** All but one of the following integrals are improper. Which integral is not improper? For each integral that *is* improper, identify the "trouble spot."

   (a) $\int_0^1 \frac{1}{x^2}\, dx$

   (b) $\int_1^2 \frac{1}{x^2}\, dx$

   (c) $\int_1^2 \frac{1}{(x-2)^2}\, dx$

   (d) $\int_0^2 \frac{1}{(x-1)^2}\, dx$

5. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Representing motion in the plane with parametric equations
2. Finding speed and velocity

Overview

Recall that motion along a straight line can be described by a position function $s(t)$, and its derivative is the velocity function $v(t) = \frac{ds}{dt}$. Velocity can be positive or negative; the sign indicates the direction of motion. Speed, on the other hand, is always positive, and it is given by the magnitude (absolute value) of the velocity.

Motion in the plane can be described by a pair of functions: $x(t)$ and $y(t)$, representing the x-coordinate and y-coordinate of the position at time t, respectively. The equations for x and y in terms of t are called **parametric equations** because they express x and y in terms of a common **parameter**, namely t.

The **velocity** of an object moving in the plane is also represented by a pair of functions: the velocity in the x-direction is $v_x(t) = \frac{dx}{dt}$ and the velocity in the y-direction is $v_y(t) = \frac{dy}{dt}$. The velocity vector is a way of expressing both of these velocities simultaneously. **Speed** simply describes how fast an object moves along the direction of its motion; it is the magnitude (length) of the velocity vector.

Assignments

1. Reading Assignment

Read the first part of Section 4.8, up to and including Example 8. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.8 # 1, 7, 9, 21, 22, 31, 49

3. Practice Problems and Quality Solution

Practice: 4.8 # 3, 5, 23, 30, 39, 52
Quality Solution: 4.8 # 26

Math 114, Parametric Equations (Section 4.8-I)

Name: ______________________________ **Section:** __________

Read the first part of Section 4.8, up to and including Example 8. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Circular and Linear Motion.** Reread Examples 1, 3, and 5.

   (a) Describe the motion of the particle whose x and y coordinates at time t are given by the equations $x = \cos(t/2)$ and $y = \sin(t/2)$.

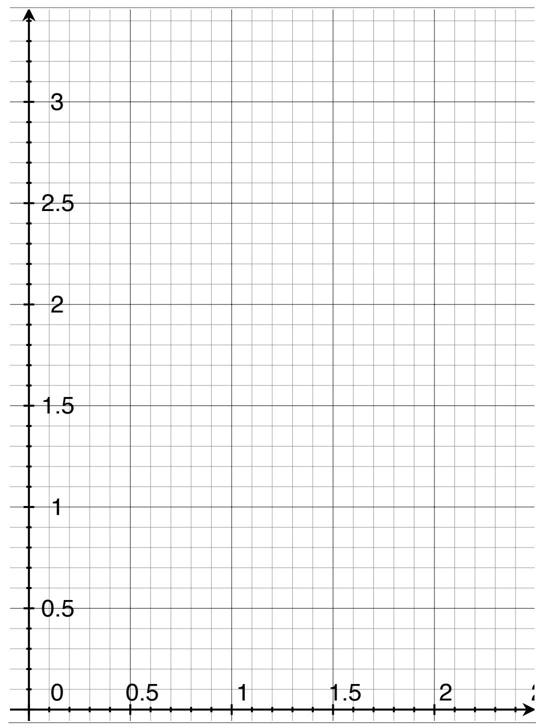
   (b) Write parametric equations for a particle moving with constant speed along a straight line from $(1, 2)$ to $(-1, 3)$. What is the slope of this line?

2. The motion of a particle in the xy-plane is described in terms of time, t, by

$$x = \ln(t) \qquad y = \sqrt{t} \qquad t \geq 1$$

(a) Make a table of x and y coordinates for the curve along which the particle moves. (Round to one decimal place.) Sketch a graph of the curve by plotting the points in your table. Indicate with an arrow the direction in which the curve is traced as t increases.

t	x	y
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		



(b) Find the velocity in the x-direction, the velocity in the y-direction, and the speed when $t = 1$.

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Representing a curve with parametric equations
2. Eliminating the parameter to find a Cartesian equation for a curve given by parametric equations
3. Tangent lines, slope and concavity of parametric curves
4. Area under parametric curve

Overview

Recall that the motion of a particle in the xy-plane can be described using parametric equations, which describe the x-coordinate and the y-coordinate of the particle at a given time t. The path in the xy-plane traced out by the particle over time is an example of a **parametric curve**, a curve whose coordinates are given by equations expressed in terms of a common variable called the **parameter**. The parameter is usually denoted t, suggesting time, but it is legitimate to use any variable for the parameter, and, in applications, the parameter does not necessarily represent time. Sometimes it is possible to **eliminate the parameter** to obtain an equation for the curve involving only x and y, as in Example 1, when the parametric curve given by $x = \cos t$, $y = \sin t$ was rewritten as $x^2 + y^2 = 1$.

For a curve in the xy-plane, the **slope of a tangent line** (assuming there is a well-defined tangent line!) to the curve at the point (a, b) is $\frac{dy}{dx}|_{(a,b)}$. We use the Chain Rule to find $\frac{dy}{dx}$ for a curve in parametric form.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \qquad (\text{if } x'(t) \neq 0)$$

Note that this will give us a formula for $\frac{dy}{dx}$ in terms of t. Let $m(t)$ be $\frac{dy}{dx}$ as a function of t. Then, to determine **concavity**, we need $\frac{d^2y}{dx^2}$, which is the derivative of m with respect to x:

$$\frac{d^2y}{dx^2} = \frac{dm}{dx} = \frac{m'(t)}{x'(t)} \qquad (\text{if } x'(t) \neq 0)$$

Since $dx = x'(t)dt$, the **signed area** between a parametric curve and the x-axis from $x(\alpha)$ to $x(\beta)$ is:

$$\int_{x(\alpha)}^{x(\beta)} y\, dx \;=\; \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt$$

as long as the curve is traversed exactly once, from left to right, as t increases from α to β.

Assignments

1. Reading Assignment

Read the second part of Section 4.8, **as well as the hand-out on finding area under a parametric curve**. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.8 # 27, 45, 50, 54, and one additional problem (see below)

3. Practice Problems and Quality Solution

Practice: 4.8 # 29, 53, 56, and one additional problem (see below)
Quality Solution: 4.8 # 28

Areas Enclosed by Parametric Curves

Suppose we have a parametric curve lying above the x-axis and the curve is traversed exactly once, from left to right, as the parameter moves from $t = \alpha$ to $t = \beta$. The vertical distance from the curve to the x-axis is given by $y(t)$ and differential is $dx = x'(t)dt$. Thus the area between the curve and the x-axis, from $x(\alpha)$ to $x(\beta)$ is:

$$A = \int_{x(\alpha)}^{x(\beta)} y\,dx = \int_{\alpha}^{\beta} y(t)\,x'(t)\,dt$$

Example 1. Find the area between the parametric curve $x = t - 1$, $y = t^2 + t$ and the x-axis from $x = 0$ to $x = 1$.

> Notice that as t increases, $x = t - 1$ also increases, so as "time" moves forward, the curve is traversed from left to right. The parameter values corresponding to $x = 0$ and $x = 1$ are $t = 1$ and $t = 2$, respectively. Notice that y is positive when $1 \le t \le 2$. Thus the area between the curve and the x-axis from $x = 0$ to $x = 1$ is:
>
> $$A = \int_1^2 y(t)x'(t)\,dt = \int_1^2 (t^2 + t)(1)dt = \int_1^2 t^2 + t\,dt = (\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2)\Big|_1^3 = \tfrac{23}{6}$$
>
> To check our work, we can eliminate the parameter and integrate with respect to x. Since $x = t - 1$, $t = x + 1$, and $y = (x + 1)^2 + (x + 1) = x^2 + 3x + 2$. Thus the area is
>
> $$A = \int_0^1 x^2 + 3x + 2\,dx = (\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x)\Big|_0^1 = \tfrac{23}{6}$$

Example 2. Find the area enclosed by the parametric curve $x = e^t$, $y = 1 - t^2$ and the x-axis.

> Notice that as t increases, $x = e^t$ also increases, so as "time" moves forward, the curve is traversed from left to right. Also notice that the x-intercepts of the curve will occur when $y = 0$, namely when $t = -1$ and $t = 1$, and when t is between -1 and 1, y is positive. Thus the area enclosed by the curve and the x-axis is given by the integral
>
> $$A = \int_{-1}^1 y(t)\,x'(t)\,dt = \int_{-1}^1 (1 - t^2)e^t\,dt$$
>
> We can evaluate this integral using IBP. Let $u = 1 - t^2$ and $dv = e^t dt$. Then $du = -2tdt$ and $v = e^t$. Thus:
>
> $$\int (1 - t^2)e^t\,dt = (1 - t^2)e^t - \int (e^t)(-2t)\,dt = (1 - t^2)e^t + 2\int te^t\,dt$$
>
> We need to use IBP again: with $u = t$, $dv = e^t dt$, we get $du = dt$ and $v = e^t$, so
>
> $$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C = (t - 1)e^t + C$$
>
> Substituting back into the original integral, we get
>
> $$\int (1 - t^2)e^t\,dt = (1 - t^2)e^t + 2\int te^t\,dt = (1 - t^2)e^t + 2(t - 1)e^t + C = -(1 - 2t + t^2)e^t + C$$
>
> So we can conclude
>
> $$A = \int_{-1}^1 (1 - t^2)e^t\,dt = -(1 - 2t + t^2)e^t\Big|_{-1}^1 = 0 + 4e^{-1} = 4/e$$

To check our work, let's eliminate a parameter and find the area again. To eliminate the parameter, notice that $x = e^t$ implies $t = \ln x$. Thus $y = 1 - (\ln x)^2$. This curve will have x-intercepts when $(\ln x)^2 = 1$. Solving this equation for x gives:

$$(\ln x)^2 = 1 \quad \Longleftrightarrow \quad \ln x = \pm 1 \quad \Longleftrightarrow \quad x = e^{-1} \text{ or } e^1 \quad \Longleftrightarrow \quad x = 1/e \text{ or } e$$

Since $1 - (\ln x)^2$ is positive between $x = 1/e$ and $x = e$, the area enclosed by the curve and the x-axis is:

$$A \;=\; \int_{1/e}^{e} 1 - (\ln x)^2\,dx \;=\; \int_{1/e}^{e} 1\,dx \;-\; \int_{1/e}^{e} (\ln x)^2\,dx \;=\; (e - 1/e) \;-\; \int_{1/e}^{e} (\ln x)^2\,dx$$

We use IBP to evaluate this integral, with $u = (\ln x)^2$ and $dv = dx$. Then $du = \frac{2\ln x}{x}$ and $v = x$, so

$$\int (\ln x)^2\,dx \;=\; x(\ln x)^2 \;-\; \int x\left(\frac{2\ln x}{x}\right)\,dx \;=\; x(\ln x)^2 \;-\; 2\int \ln x\,dx$$

Using IBP again, with $u = \ln x$, $dv = dx$, we have $du = \frac{1}{x}\,dx$ and $v = x$, so

$$\int \ln x\,dx \;=\; x\ln x \;-\; \int x\left(\frac{1}{x}\right)\,dx \;=\; x\ln x \;-\; \int dx \;=\; x\ln x - x + C \;= x(\ln x - 1) + C$$

Substituting back into the original integral, we have

$$\int (\ln x)^2\,dx \;=\; x(\ln x)^2 \;-\; 2x(\ln x - 1) + C \;=\; x\big((\ln x)^2 - 2\ln x + 2\big) + C$$

Therefore

$$\int_{1/e}^{e} (\ln x)^2\,dx \;=\; x\big((\ln x)^2 - 2\ln x + 2\big)\Big|_{1/e}^{e} \;=\; e(1 - 2 + 2) - (1/e)(1 + 2 + 2) \;=\; e - 5/e$$

And the area is

$$A \;=\; \int_{1/e}^{e} 1 - (\ln x)^2\,dx \;=\; (e - 1/e) \;-\; \int_{1/e}^{e} (\ln x)^2\,dx \;\; (e - 1/e) - (e - 5/e) \;=\; 4/e$$

This agrees with our answer from above.

Additional Discussion Problem.

Sketch the curve $x = 3\cos\theta$, $y = 4\sin\theta$, $0 \le \theta \le 2\pi$, and find the area that it encloses. (Hint: Use symmetry. Divide the area into four equal pieces and find the area of one piece, then multiply by four to get the total area.)

Additional Practice Problem.

Sketch the curve $x = 9 + e^t$, $y = t - t^2$, and find the area enclosed by the curve and the x-axis.

Math 114, Parametric Curves (Section 4.8-II)

Name: ______________________________ **Section:** __________

Read the second part of Section 4.8, **as well as the hand-out on finding area under a parametric curve**. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Slope and Concavity.** Reread Example 9 and the discussion of slope and concavity, p 255-256.

   Consider the parametric curve in Example 9: $x = t^3$, $y = 2t$.

   (a) According to Example 9, what are $\frac{dy}{dt}$ and $\frac{dx}{dt}$?

   (b) According to Example 9, what are the parametric equations for the tangent line to the curve at the point $(1, 2)$? What is the slope of this line? (You may use the formula on page 251.)

   (c) Using the equation for the slope of a parametric curve on page 255, what is $\frac{dy}{dx}$ as a function of t? What is the slope of the parametric curve when $t = 1$?

   (d) Find the point on the parametric curve corresponding to $t = 1$. Use this point and the slope you found in (d) to find an equation for the tangent line to the curve at when $t = 1$.

   (e) Let $m(t)$ be the slope of the parametric curve as a function of t (the one you found in (d)). What is $m'(t)$? To find $\frac{d^2y}{dx^2}$, divide $m'(t)$ by $x'(t)$. Is the curve concave up or concave down when $t = 1$?

2. **Area.** Consider the parametric curve in Example 9: $x = t^3$, $y = 2t$. We will find the area between the curve and the x-axis from $x = 0$ to $x = 8$.

   (a) Notice that when t increases, $x = t^3$ also increases, so the curve is traversed from left to right. What are the t-values corresponding to $x = 0$ and $x = 8$? Call them α and β, respectively.

   (b) Check that the curve is above the x-axis ($y \geq 0$) when t is between α and β.

   (c) Using the fact that $dx = x'(t)dt$, find dx.

   (d) Find the area between the curve and the x-axis from $x = 0$ and $x = 8$ using the area formula:

$$A \quad = \quad \int_{x(\alpha)}^{x(\beta)} y\, dx \quad = \quad \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt$$

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Area between two curves
2. Average value of a continuous function on an interval

Overview

Area Recall that we use a definite integral to find the (signed) area between a curve and the x-axis. If $f(x) \geq 0$ on an interval $[a, b]$, then the definite integral gives a literal area:

$$\text{(area between } f(x) \text{ and } x\text{-axis from } x = a \text{ to } x = b) \quad = \quad \int_a^b f(x)\,dx$$

Similarly, if a function $f(x) \geq g(x)$ on an interval $[a, b]$, the **area between the two curves from** $x = a$ **to** $x = b$ is obtained by subtracting the smaller area from the greater area:

$$\text{(area between } f(x) \text{ and } g(x) \text{ from } x = a \text{ to } x = b) \quad = \quad \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \quad = \quad \int_a^b \big(f(x) - g(x)\big)\,dx$$

When two continuous curves cross each other more than once, we can find the area of the region (or regions) between the two curves using this idea, once we have determined (a) the intersection points of the two curves and (b) which curve lies above the other in each region.

Average Value Another application of the integral is finding the average value of a quantity that changes in a continuous way. For example, to estimate the average temperature in a given city over the course of a year, we might take the average of the temperatures recorded at noon on the first day of each month: add up these temperatures and divide by twelve. We could improve our estimate by using weekly, daily, even hourly temperature recordings. The sums that are used to compute these averages are Riemann sums; taking a limit as the number of recordings goes to infinity gives a definite integral. See the discussion on page 304 for the details of the derivation of the average value formula:

$$\text{(average value of } f(x) \text{ on } [a, b]) \quad = \quad \frac{1}{b - a} \int_a^b f(x)\,dx$$

Assignments

1. Reading Assignment

Read "Area Between Curves" and "The Definite Integral as an Average" in Section 5.4. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

5.4 # 7, 13, 18, 41

3. Practice Problems and Quality Solution

Practice: 5.4 # 8, 15, 20*, 32
Quality Solution: 5.4 # 42*
*For # 20, do not try to find the intersection point exactly; estimate it by zooming in on the graphs. For #42, use this fact: Since $a^t = e^{\ln(a^t)} = e^{(\ln a)t}$, $P = 112(1.011)^t = 112e^{(\ln(1.011))t}$.

Math 114, Area Between Two Curves; Average Value (Section 5.4)

Name: ______________________________ **Section:** __________

Read "Area Between Curves" and "The Definite Integral as an Average" in Section 5.4. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Area Between Two Curves.** Reread Example 3. Now consider the region below the curve $y = 5x - x^2$ and above the curve $y = x$.

   (a) Where do the two curves cross? (Solve the equation $x = 5x - x^2$ for x.)

   (b) Sketch the region.

   (c) Find the area of this region by subtracting one definite integral from another.

2. **Average Value**

   (a) Reread Example 6. If $V(t)$ represents the value, in dollars, of a Tiffany lamp t years after 1975, write an expression (involving an integral) that represents the average value of the lamp over the period 1975-2010.

(b) Find the average value of the function $f(x) = 5x - x^2$ over the interval $[0, 5]$. Sketch a graph of the function on this interval. Does your answer make sense?

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Approximating areas and volumes with Riemann sums of "slices"
2. Calculating area and volume exactly using definite integrals

Overview

Recall that the definite integral can be used to find the area under a curve. The area is first approximated by a Riemann sum, which is the sum of the areas of rectangles, whose heights are given by the y-values of the curve and whose widths are Δx, small changes in x. Taking a limit as the number of rectangles goes to infinity gives the exact area.

We generalize this procedure to find **areas of various regions** in the plane: we slice the region into thin strips, approximate each strip by a rectangle, add up the areas of the rectangles, and take a limit as the number of rectangles goes to infintity. If we can represent this sum as a Riemann sum, then the limit is a definite integral, and we can try to find the area using an antiderivative.

In particular, to find the **area of a region enclosed by two curves** in the plane, we usually slice the region vertically (in which case the thickness of the strips is Δx) or horizontally (in which case the thickness of each strip is Δy). When slicing vertically, the heights of the approximating rectangles will be given by the vertical distance between the two curves, which can be computed by subtracting the height of the bottom curve from the height of the top curve: $h(x) = \text{Top}(x) - \text{Bottom}(x)$. When slicing horizontally, the widths of the approximating rectangles wil be given by the horizontal distance betwen the two curves, which can be computed by subtracting the left curve from the right curve: $w(y) = \text{Right}(y) - \text{Left}(y)$.

To find the **volume of a solid**, we slice the solid into thin slices, whose crossectional area is known from geometry (for example, the area of a circle, rectangle, or triangle.) We then approximate the volume of each slice by multiplying the crossectional area by the thickness, add up these volumes, and take a limit as the number of slices goes to infinity. Again, if we can represent the sum as a Riemann sum, then the limit is a definite integral and we can try to find the volume using an antiderivative.

Assignments

1. Reading Assignment

Read Section 8.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.1 # 7, 11, 12, 14, 27, 42

3. Practice Problems and Quality Solution

Practice: 8.1 # 2, 3, 17, 28, 29, 41
Quality Solution: 8.1 # 30

Math 114, Areas and Volumes (Section 8.1)

Name: ______________________________ **Section:** __________

Read Section 8.1. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Area: Vertical and Horizontal Slices.** Reread Example 1. Now consider the triangle in Exercises 1 and 4, in the exercises at the end of the section.

   (a) Use the formula $A = \frac{1}{2}bh$ to find the area of this triangle.

   (b) Find the area of the triangle using vertical slices, as in Exercise 1. Note that the (very small) width of each slice is Δx and the height of each slice is $h(x) = 2x$.

   (c) Find the area of the triangle using horizontal slices, as in Exercise 4. Note that the (very small) height of each slice is Δy. Use similar triangles to find the width w of each slice in terms of y. You will integrate $w(y)$ with respect to y, with y ranging from $y = 0$ to $y = 6$.

2. **Volume.** Reread the paragraphs between Example 2 and Example 3, which show how to find the volume of a cone.

   (a) In Example 3, we slice the cone into horizontal slices. Why not slice the cone vertically, as shown in Figure 8.5?

   (b) When we slice the cone horizontally, each slice is approximately a disk. We will find the volume of each disk and add up the volumes of each disk to approximate the volume of the cone.

   What is the thickness of each disk?

   What is the radius of each disk?

   What is volume of each disk?

   (c) What is the Riemann sum that approximates the volume of the cone? What is the integral that expresses the exact volume of the cone? (Don't evaluate the integral; just write it down.)

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Finding volumes of revolution by slicing solid into disks or washers
2. Volumes of solids constructed by standing squares, semicircles, or triangles on edge in a planar region.

Overview

Recall our strategy for finding the volume of a solid: we slice the solid into thin slices, whose crossectional area is known from geometry, approximate the volume of each slice, add up these volumes, and take a limit as the number of slices goes to infinity. If we can represent the sum as a Riemann sum, then the limit is a definite integral and we can try to find the volume using an antiderivative.

We now focus our attention on finding volumes of solids constructed from planar regions in a couple specific ways.

Solids of revolution are constructed by rotating a planar region about some axis. When we slice a solid of revolution perpendicularly to the axis of rotation, each slice can be approximated by a circular disk or washer (a circular disk with a hole in the center), whose volume is easy to compute. See Examples 1-3.

Given a planar region, we may also construct a solid by **standing squares, semicircles, or triangles on edge** in this region. See Example 4.

Assignments

1. Reading Assignment

Read Section 8.2 up to but not including the part about arc length. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.2 # 7, 13, 44-48

3. Practice Problems and Quality Solution

Practice: 8.2 # 8, 11, 50-54, 62
Quality Solution: 8.2 # 14

Math 114, Applications to Geometry: Volumes by Slicing (Section 8.2-I)

Name: ______________________________ **Section:** __________

Read Section 8.2 up to but not including the part about arc length. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Rotating Around the x-axis.** Reread Example 1.

   (a) Sketch the planar region described in Example 1. Then, in a separate drawing, sketch the solid obtained by revolving this region around the x-axis.

   (b) We slice the solid perpendicularly to the axis of rotation. What is the shape of a typical slice? (A circular disk or a circular washer?) What is the volume of a typical slice?

   (c) Write (but do not evaluate) the integral representing the volume of the solid.

   (d) Now try Exercise 1 in the exercises at the end of the section. Include a sketch of the solid.

2. **Rotating Around Another Horizontal Line.** Reread Example 3.

   (a) Sketch the planar region described in Example 3 and the line $y = 3$. Then, in a separate drawing, sketch the solid obtained by revolving this region around the line $y = 3$.

   (b) We slice the solid perpendicularly to the axis of rotation. What is the shape of a typical slice? (A circular disk or a circular washer?) What is the volume of a typical slice?

   (c) Write (but do not evaluate) the integral representing the volume of the solid.

3. **Another Way to Construct Solids.** Reread Example 4

   (a) What shape is the typical slice in this example? What is the volume of a typical slice?

   (b) Write (but do not evaluate) the integral representing the volume of the solid.

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Arc length formulas
2. Using *Mathematica* for numerical integration

Overview

We can use a definite integral to express the length of a curve: we divide the curve into short segments, approximate the length of each segment by assuming the segment is straight, add up these approximate lengths to obtain a Riemann sum, and then take a limit as the number of segments approaches infinity.

As usual, *if* it is possible to find an antiderivative for the integrand, we can use the Fundamental Theorem of Calculus to evaluate the integral and find the arc length of the curve exactly. However, it turns out that, in arc length problems, frequently the integrand does *not* have an elementary antiderivative; numerical methods are needed to approximate the integral. We can use *Mathematica* to plot the curves and to estimate their arc lengths numerically.

The curve $y = x^3$ from $x = 0$ to $x = 5$ in Example 5 can be plotted with the *Mathematica* `Plot` command:

```
Plot[x^3, {x, 0, 5}]
```

The integral for the arc length is $\int_0^5 \sqrt{1 + (3x^2)^2}\, dx$. It can be estimated numerically in *Mathematica*:

```
NIntegrate[Sqrt[1+(3x^2)^2], {x, 0, 5}]
```

The parametric curve $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ in Example 6 can be plotted with `ParametricPlot`:

```
ParametricPlot[{2*Cos[t], Sin[t]}, {t, 0, 2*Pi}]
```

The integral for its arc length is $\int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\, dt$ and can be estimated numerically by:

```
NIntegrate[Sqrt[4(Sin[t])^2 + (Cos[t])^2], {t, 0, 2*Pi}]
```

Assignments

1. Reading Assignment

Read the part of Section 8.2 about arclength. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.2 # 19*, 21*, 23, 25, 27*, 90**
*Use *Mathematica* to plot the curve and to estimate the arc length using numerical integration. Use the `Log` command for the natural log in *Mathematica*.
**Hints: You will need to use the fact that $\sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2} = \frac{1}{2}(e^x + e^{-x})$. Use algebra to show this is true. Also, at the end of the problem, you need to solve $e^b - e^{-b} = 10$ for b. Use a graph to estimate b instead of trying to find b exactly.

3. Practice Problems and Quality Solution

Practice: 8.2 # 20*,22, 26, 78
Quality Solution: 8.2 # 28*
*Use *Mathematica* to plot the curve and estimate the arc length using numerical integration. Absolute value is `Abs`.

Math 114, Applications to Geometry: Arc Length (Section 8.2-II)

Name: ______________________________ **Section:** ________

Read the part of Section 8.2 about arclength. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. Reread Example 5.

   (a) Use the `Plot` command in *Mathematica* to plot the curve.

   (b) What is the integral for the arc length in this example?

   (c) Use the `NIntegrate` command in *Mathematica* to estimate the arc length, and write your estimate below.

   (d) Now consider the curve $y = x^2$ from $x = 0$ to $x = 3$. Use the `Plot` command in *Mathematica* to plot the curve. Sketch the curve below.

   (e) What is the integral for the arc length of $y = x^2$ from $x = 0$ to $x = 3$?

   (f) Use the `NIntegrate` command in *Mathematica* to estimate the arc length, and write your estimate below. Does this estimate seem reasonable? Explain.

2. Reread Example 6.

   (a) Use the `ParametricPlot` command in *Mathematica* to plot the curve.

   (b) What is the integral for the arc length in this example? Use the `NIntegrate` command in *Mathematica* to estimate the arc length.

   (c) Now consider the parametric curve $x = 20 \sin t$, $y = t^2$, $0 \le t \le 2\pi$. Use the `ParametricPlot` command in *Mathematica* to plot the curve. Sketch the curve below.

   (d) What is the integral for the arc length of $x = 20 \sin t$, $y = t^2$, $0 \le t \le 2\pi$?

   (e) Use the `NIntegrate` command in *Mathematica* to estimate the arc length.

3. What struck you in reading this section? What is still unclear? What questions do you have?

Sections 8.4, 8.5, and 8.6 cover various practical applications of the integral, primarily in physics and economics. Instead of introducing this material with reading and discussion assignments, we will have **group presentations** on topics from each section.

Preparing and Giving the Presentation

1. Read and discuss the relevant section (or portion of a section) of the textbook.
2. Solve your assigned presentation problem. An asterisk indicates a challenge problem.
3. Write up your solution (only one per group) and turn it in on the day that you present.
   - Your written solution will count as a quality solution. (+1) for challenge problem.
4. Create and rehearse your presentation. (Time yourselves!)
5. Outline your presentation on the board, *before* class starts, on the day of your presentation.
6. Present your topic, in a 7-10 minute oral presentation, in which each group member speaks.
   - Explain the physics or economics background and why an integral is appropriate in this situation; then explain the solution to your assigned problem.
   - The oral presentation will count towards your grade like a quality solution.
   - Graded for: clarity of delivery, depth of conceptual explanations, correctness of content.

1. Presentation Problems

Each group of students will present one or two of the eight application topics below. I will email the class to solicit input on the assignment of groups and topics. Once you have been assigned a topic, choose one of the presentation problems for your topic, listed below. You will receive a grade for the written solution of your problem as well as for the oral presentation of your problem. If you choose a challenge problem, the written solution receives a bonus, but the oral presentation does not.

8.4	(Nov 10)	Density: 14 or 18*; Center of Mass: 22 or 26*
8.5	(Nov 13)	Work: 12 or 24*; Pressure: 26a* or 30; Energy: KE-1*; Gravitation: G-1*
8.6	(Nov 15)	Income Stream: 18 or 34*; Consumer/Producer Surplus: 36*

Note. For kinetic energy and gravitation, some outside research will be needed in order to prepare an adequate discussion of the background in the presentation. This will be taken into account in the grading of the presentation. See the back of this page for the kinetic energy and gravitation problems.

2. Practice Problems

After hearing the presentations on a given section, you will have an opportunity to start the practice problems in class. The practice problems are due at the beginning of the next class.

P 8.4	(due Nov 13)	Density: 5, 9, 13, 16, 17, 19;	Center of Mass: 8, 21, 24, 25, 27
P 8.5	(due Nov 15)	Work: 4, 5, 11, 13, 15, 17;	Force and Pressure: 7, 28, 31, 32, 33
P 8.6	(due Nov 17)	Income Stream: 10, 11, 21, 23, 25;	Consumer/Producer Surplus: 39, 41

3. Quality Solutions

Choose **two** of the following problems (not both on the same topic) to write up nicely, as quality solutions. These are due at the beginning of class **Nov 17**, which is the first class after we have finished all the presentations. *All of these are challenge problems* and will receive the (+1) bonus.

QS 8.4: Density: 32, 34 Center of Mass: 28, 30
QS 8.5: Work: 36, 38; Pressure: 34; Energy: KE-2; Gravitation: G-2 or G-3.
QS 8.6: Income Stream: 32; Consumer/Producer Surplus: 40

Kinetic Energy. The kinetic energy E of a particle of mass m moving at a speed v is $E = \frac{1}{2}mv^2$.

KE-1. **Presentation:** Find the kinetic energy of a thin uniform rod of mass 10 kg and length 6 m rotating, like a helicopter blade, about an axis perpendicular to the rod at its midpoint, with an angular velocity of 2 radians per second. (Hint: The speed v of an object traveling with angular velocity ω, in radians per unit time, in a circle of radius r is $v = r\omega$. Why does this make sense?)

KE-2. **QS:** Find the kinetic energy of a phonograph record of uniform density, mass 50 gm and radius 10 cm rotating at $33\frac{1}{3}$ revolutions per minute. (See hint in previous problem.)

Gravitation. The force F of gravitational attraction between two particles of mass m_1 and m_2 at a distance r apart is $F = Gm_1m_2/r^2$, where G is a constant called the universal gravitational constant.

G-1. **Presentation:** What is the force of gravitational attraction between a thin uniform rod of mass M and length ℓ and a particle of mass m lying in the same line as the rod at a distance a from one end? (You should get $GMm/(a(a + \ell))$ for your final answer.)

G-2. **QS:** Two long, thin, uniform rods of lengths ℓ_1 and ℓ_2 lie on a straight line with a gap between them of length a. Suppose their masses are M_1 and M_2, respectively. What is the force of gravitational attraction between the two rods? (Hint: Use the answer from the presentation problem.)

G-3. **QS:** Find the gravitational force exerted by a thin uniform ring of mass M and radius a on a particle of mass m lying on a line perpendicular to the ring through its center. Assume m is at a distance y from the center of the ring.

Main Points:

1. Mathematical meaning of sequence
2. Recursive sequences
3. Convergence of sequences
4. Monotone bounded sequences

Overview

A **sequence** is an infinite list of numbers in a definite order. The numbers in the sequence are called **terms**. One way of describing a sequence is by listing the first several terms, as in these examples:

$$1, 2, 3, 4, 5, 6, \ldots \qquad 1, -1, 1, -1, 1, -1, \ldots \qquad 1,\ 1/2,\ 1/4,\ 1/8, \ldots$$

Another way of describing a sequence is by giving a formula for the n^{th} term of the sequence. For example the three sequences above could be represented with the following three formulas:

$$a_n = n\ , \ \ n \geq 1 \qquad b_n = (-1)^n\ , \ \ n \geq 0 \qquad c_n = 1/2^n\ , \ \ n \geq 0$$

Some sequences are more easily described **recursively**. In a recursively defined sequence, the first term (or the first few terms) are given along with a formula for how to find successive terms. For example, the first sequence above could be defined recursively as: $a_1 = 1$, $a_n = a_{n-1} + 1$ for $n > 1$.

If the numbers a_n approach a specific, finite number L as $n \to \infty$, then the sequence is said to **converge**, and L is called the **limit** of the sequence. If a sequence does not have a limit, it is said to **diverge**.

Assignments

1. Reading Assignment

Read Section 9.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.1B # 16, 17, 20, 21, 24, 27, 65, 66, 70, 75

3. Practice Problems and Quality Solution

Practice 9.1A: # 4, 8, 11, 28, 29, 31, 36, 37, 39, 59
Quality Solution 9.1A: # 60

Practice 9.1B:# 19, 22, 68, 69
Quality Solution 9.1B: # 72, 73, 76

Math 114, Sequences (Section 9.1)

Name: ______________________________ **Section:** ________

Read Section 9.1. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Sequences.** Reread Examples 1 and 3. Give the first six terms for the following sequences:

   (a) $s_n = 2^n + 1$, $n \geq 1$.

   (b) $s_n = s_{n-1} + s_{n-2}$ for $n > 2$ and $s_1 = 1$, $s_2 = 1$

2. **Convergence of Sequences** What are the "two facts" that we can use to calculate the limit of a sequence, in addition to what we know about the limits of functions? (These are listed right before Example 5.)

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Mathematical meaning of series
2. Finite and infinite geometric series

Overview

Adding the terms in a sequence results in a **series**. Of course, one ought to be suspicious of whether adding infinitely many numbers can actually result in a finite number. Sometimes it does; sometimes it doesn't. A **convergent** series is one that does have a finite sum; a **divergent** series does not have a finite sum. We will use limits to make the notion of convergence precise, as we did in our discussion of improper integrals.

We may try to approximate the sum (if it exists) of an infinite series $a_0 + a_1 + a_2 + \dots$ by adding up a large (but finite) number of terms. Adding up the first n terms results in the nth **partial sum**: $S_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}$. In hopes of obtaining better and better approximations, we add more and more terms (i.e. we let n approach infinity). However, the partial sums may grow unboundedly (or exhibit other wild behavior); in such cases the series diverges. If, in contrast, the partial sums approach a specific finite number, as we add more and more terms, the series converges and the sum S of the series is defined to be the limit of the partial sums: $S = \lim\limits_{n\to\infty} S_n$.

One of the simplest kinds of series is a **geometric series**, one in which the ratio of successive terms is a constant, called the **common ratio**. For example, $3 + 6 + 12 + 24 + 48 + \dots$ is a geometric series with common ratio 2, and $3 + 3/2 + 3/4 + 3/8 + 3/16 + \dots$ is geometric with common ratio 1/2. It is clear that the first geometric series diverges; that the second series converges is perhaps not quite as obvious, but still relatively straightforward. Imagine that there are six cupcakes left after your birthday party, so you eat three yourself, then give half of what's left to your friend (3/2 cupcakes). Your friend eats half of what you gave her and gives the rest (3/4 a cupcake) to you, who again eat half and give the rest to her (3/8 a cupcake). If you continue in this way indefinitely, how much will the two of you eat? Well, certainly not an infinite amount, because you only started with six cupcakes!

It turns out that a geometric series diverges when the absolute value of the common ratio is greater than or equal to one (again, this is fairly obvious) and converges when the absolute value of the common ratio is less than one. This can be proven rigorously, by looking at the limit of partial sums.

Assignments

1. Reading Assignment

Read Section 9.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.2 # 1, 2, 8, 10, 24, 28, 29, 55, 57

3. Practice Problems and Quality Solution

Practice 9.2: # 3, 4, 6, 9, 12, 26, 46, 56
Quality Solution 9.2: # 30

Math 114, Geometric Series (Section 9.2)

Name: ____ **Section:** ________

Read Section 9.2. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Sum of a Finite Geometric Series.** Reread the part entitled "Sum of a Finite Geometric Series." Now consider the finite geometric series $5 + 5/3 + 5/9 + 5/27$.

   (a) What is the first term of this geometric series?

   $a =$

   (b) What is the common ratio for this geometric series?

   $x =$

   (c) How many terms are there in this geometric series?

   $n =$

   (d) Calculate the sum of this finite geometric series, using the formula in the box.

2. **Sum of an Infinite Geometric Series.** Reread the part entitled "Sum of an Infinite Geometric Series." Consider the infinite geometric series: $5 + 5/3 + 5/9 + 5/27 + \ldots$

   (a) Does the series converge or diverge? How can you tell, without taking a limit?

   (b) Give a formula for S_n, the nth partial sum of this series.

   (c) Calculate the sum of the infinite series using the formula in the box.

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Convergence properties of series
2. Comparison with improper integrals
3. The harmonic series and p-series

Overview

In the previous section, we looked at a special family of infinite series: infinite geometric series; we used the formula for the sum of a finite geometric series to find the sum of a convergent infinite geometric series, using the limit of partial sums. We now expand our view to look at convergence and divergence of series more generally.

Recall that an infinite series $a_0 + a_1 + a_2 + \dots$ is convergent if the limit of partial sums is finite. In this case the sum of the series S is the limit of partial sums: $S = \lim_{n\to\infty} S_n$, where $S_n = a_0 + a_1 + \cdots + a_{n-1}$. A series is divergent if it does not converge.

Theorem 9.2 lists several straightforward **convergence properties** of series. Make sure to include the full statement of this theorem in your notes.

We next use what we know about improper integrals to discuss the convergence and divergence of infinite series. The Riemann sum of an improper integral is an infinite series; the convergence or divergence of the integral can help us determine the convergence or divergence of the series. This idea is made precise in **the integral test**. The integral test allows us to determine convergence/divergence in an important family of examples: the ***p*-series**, which includes as a special case the **harmonic series.**

Assignments

1. Reading Assignment

Read Section 9.3. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.3 # 1, 4, 23, 25, 33, 37
Hints: For 23, 25, and 33, use Theorem 9.2. For 33, also use the fact that $\ln(2^n) = \ln(2) \cdot n$.

3. Practice Problems and Quality Solution

Practice 9.3: # 2, 5, 6, 24, 34, 35
Quality Solution 9.3: # 38, 8*

Math 114, Convergence of Series (Section 9.3)

Name: ______________________ **Section:** ________

Read Section 9.3. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **The Limit of Partial Sums.** Reread Example 1.

   (a) What is a_n in this example? S_n?

   $a_n =$ $S_n =$

   (b) What is $\lim_{n\to\infty} a_n$? $\lim_{n\to\infty} S_n$?

   $\lim_{n\to\infty} a_n =$ $\lim_{n\to\infty} S_n =$

   (c) What is the sum of the series?

   $S =$

2. **Convergence Properties of Series.** Reread Theorem 9.2.

   (a) Use Theorem 9.2.1 to explain why $\sum_{n=1}^{\infty} \left((\frac{1}{2})^{n-1} + (\frac{1}{3})^{n-1}\right)$ is convergent and to find the sum.

   Hint: Notice $\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$ is geometric; the first term is $a = 1$ and the common ratio $x = \frac{1}{2}$.

   (b) Use Theorem 9.2.3 to explain why $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

   (c) Use (b) and Theorem 9.2.4 to explain why $\sum_{n=1}^{\infty} \frac{5n}{n+1}$ diverges.

3. **An Important Family of Examples.** Reread Examples 3, 4, and 5.

   (a) What is the name of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$? Does it converge or diverge?

   (b) State whether the following series converge or diverge. (Use Theorem 9.2 and Example 5.)

$$\sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\sum_{n=1}^{\infty} \frac{5}{n^6}$$

$$\sum_{n=1}^{\infty} \left(\frac{2}{n^6} + \frac{3}{n^2}\right)$$

$$\sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$$

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Testing series for convergence
2. The Ratio Test

Overview

Recall that the sum of a convergent infinite series is the limit of its partial sums; if the limit of partial sums does not exist, the series is divergent and does not have a finite sum. In many cases, finding a fomula for the nth partial sum is impractical, but, fortunately, there are ways to determine whether or not a series converges without having to compute the limit of partial sums explicitly. An example we have discussed is the Integral Test; several other tests are discussed in Section 9.4. We will focus on the **Ratio Test**.

Recall that a geometric series converges if the ratio of successive terms (which is a constant, called the common ratio) has absolute value less than one. For a series $\sum a_n$ that is not geometric, the ratio a_{n+1}/a_n of successive terms will not be a constant, but if the absolute value of the ratio *approaches* a constant *less than one* as n increases, the series converges. This is the idea behind the Ratio Test.

Assignments

1. Reading Assignment

Read the part of Section 9.4 about the Ratio Test. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.4 # 16, 17, 21, 55, 64

3. Practice Problems and Quality Solution

Practice 9.4: # 15, 19, 20, 56, 70
Quality Solution 9.4: # 18

Math 114, The Ratio Test (Section 9.4)

Name: ______________________ **Section:** ________

Read the part of Section 9.4 entitled "Comparison with a Geometric Series: The Ratio Test." Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. Reread Example 6, in which $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$.

   (a) What is a_n in this example? What is $|a_{n+1}|/|a_n|$?

   $a_n =$ $\frac{|a_{n+1}|}{|a_n|} =$

   (b) What is $\lim_{n\to\infty} a_n$? $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$?

   $\lim_{n\to\infty} a_n =$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} =$

   (c) What does the Ratio Test allow use to conclude about this series?

2. Reread Example 7. Consider the first series in this example, $\sum_{n=1}^{\infty} \frac{1}{n}$.

   (a) What is a_n for this series? What is $|a_{n+1}|/|a_n|$?

   $a_n =$ $\frac{|a_{n+1}|}{|a_n|} =$

   (b) What is $\lim_{n\to\infty} a_n$? $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$?

   $\lim_{n\to\infty} a_n =$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} =$

   (c) What does the Ratio Test allow use to conclude about this series? What do we know about this series from Section 9.3?

3. Now consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

   (a) What is $|a_{n+1}|/|a_n|$?

   Hints: Here are a few things to keep in mind: (1) dividing by a fraction is the same as multiplying by the reciprocal, (2) for any number a, we know that $a^{n+1}/a^n = a$, and (3) as in Example 6, the ratio of successive factorials cancels nicely: $n!/(n+1)! = 1/(n+1)$.

   (b) What is $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$?

   (c) What does the Ratio Test allow use to conclude about this series?

4. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Power series as "polynomials with infinitely many terms"
2. Domain of power series, radius of convergence
3. Using geometric series to find rational function for power series

Overview

Not all functions that turn out to be interesting or useful for applications can be described in terms of familiar functions. For example, the function e^{-x^2} is used in probability, but its antiderivative is not elementary: it cannot be expressed in terms of familiar functions. It is called the "error function," sometimes denoted erf. Another example is the "Bessel functions," which are used to model electromagnetic waves, heat conduction, and vibrating membranes.

One way to represent such functions is as **power series**, which can be thought of as polynomials with infinitely many terms. (Of course, there is the question of convergence.) In general, a power series about $x = a$ is of the form:

$$C_0 \ + \ C_1(x-a) \ + \ C_2(x-a)^2 \ + \ C_3(x-a)^3 \ + \ \dots$$

The **domain** of a power series consists of all real numbers x for which the series converges. In general, the domain of a power series will be an interval centered around $x = a$. The distance from zero to either endpoint is called the **radius of convergence**. The endpoints of the interval may or may not be included. We typically use the Ratio Test to determine the radius of convergence. Usually, further work is needed to determine convergence at the endpoints.

Two extreme cases are worth discussing separately: (1) if the power series converges only at $x = a$, the domain is simply $\{a\}$ and the radius of convergence is $R = 0$ and (2) if the power series converges for all x, the domain is $(-\infty, \infty)$ and we say that the radius of convergence is $R = \infty$.

The geometric series $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and diverges otherwise. Viewed as a power series, it is a function with domain $(-1, 1)$. We can find a rational function that agrees with the power series on its domain using the formula for the sum of a geometric series:

$$\frac{1}{1-x} \ = \ \sum_{n=0}^{\infty} x^n \ = \ 1 \ + \ x \ + \ x^2 \ + \ x^3 \ + \ \dots$$

We can use this fact to find power series representations of some functions that are similar to this one. For example, with first term 3 and common ratio $4x$,

$$\frac{3}{1-(4x)} \ = \ \sum_{n=0}^{\infty} 3 \cdot (4x)^n \ = \ \sum_{n=0}^{\infty} (3 \cdot 4^n)\, x^n \ = \ 3 \ + \ 12x \ + \ 48x^3 \ + \ \dots$$

This converges when $|4x| < 1$, or $|x| < \frac{1}{4}$. Thus the domain of the power series is $(-\frac{1}{4}, \frac{1}{4})$.

Assignments

1. Reading Assignment

Read Section 9.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.5 # 5, 7, 11, 17, 18, 19, 35, 47

3. Practice Problems and Quality Solution

Practice 9.5: # 12, 13, 24, 36, 49
Quality Solution 9.5: # 16

Math 114, Power Series and Radius of Convergence (Section 9.5)

Name: __________________________________ **Section:** ________

Read Section 9.5, focusing on the ideas of power series and radius of convergence. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Geometric Power Series.** Reread Example 2.

   (a) Determine whether the given power series converges for $x = 1$.

   (b) Notice that this is a geometric series with first term 1 and common ratio $\frac{x}{2}$. Use what you know about the convergence of geometric series to determine for what values of x the series converges.

   (c) Use the formula for the sum of an infinite geometric series to write this power series as a rational function.

2. **Non-geometric Power Series.** Reread Examples 3 and 4. Notice that these power series are *not* geometric, so we need to use the Ratio Test to determine for what values of x the series converges.

   (a) In Example 3, what is a_n? What is $|a_{n+1}|/|a_n|$?

   (b) Continuing with Example 3, what is $\lim_{n\to\infty} |a_{n+1}|/|a_n|$?

(c) Continuing with Example 3, for what values of x is $\lim_{n\to\infty} |a_{n+1}|/|a_n| < 1$? (In other words, for what values of x does the series converge?)

(d) Now look at Example 4. In this example, what is a_n? What is $|a_{n+1}|/|a_n|$?

(e) Continuing with Example 4, what is $\lim_{n\to\infty} |a_{n+1}|/|a_n|$?

(f) Continuing with Example 4, for what values of x is $\lim_{n\to\infty} |a_{n+1}|/|a_n| < 1$? (In other words, for what values of x does the series converge?)

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Recall the notion of a Taylor series
2. Binomial series expansion
3. Finding new Taylor series by substitution

Overview

Recall our study of **Taylor polynomials** at the beginning of the semester. The first three Taylor polynomials for a function $f(x)$ centered at a given point $x = a$ are simply the constant approximation: $T_0(x) = f(a)$, the linear approximation $T_1(x) = f(a) + f'(a)(x - a)$, and the quadratic approximation $T_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$. In general the Taylor coefficient C_n for the $(x - a)^n$ term in the Taylor polynomial is $C_n = f^{(n)}(a)/n!$, where $f^{(n)}$ refers to the nth derivative of f, so

$$T_n(x) \;=\; f(a) \,+\, f'(a)(x-a) \,+\, \frac{f''(a)}{2!}(x-a)^2 \,+\, \frac{f'''(a)}{3!}(x-a)^3 \,+\, \dots \,+\, \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Each successive Taylor polynomial gives a better approximation for $f(x)$ near $x = a$. We can describe the family of *all* Taylor polynomials for a given function centered at a given point using **Taylor series**.

Now that we have studied infinite series and power series in particular, we are in a position to discuss Taylor series in more depth and with more rigor. In particular, now that we have a precise understanding of convergence, we can find the radius of convergence of a Taylor series analytically (using the Ratio Test).

We take this as an opportunity to review the Taylor series for e^x, $\sin x$, and $\cos x$ and to discuss an important family of examples from Section 10.2 that we did not discuss at the beginning of the semester: the family of **binomial series** expansions, namely Taylor series for functions of the form $f(x) = (1 + x)^p$, for some real number p. In the first part of Section 10.3, we discuss how to find **new Taylor series by substitution**.

Assignments

1. Reading Assignment

Read Section 10.2 and the beginning up Section 10.3, up through Example 1 on page 553. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

10.2 # 1, 4, 39, 61;   10.3 # 1, 4, 61

3. Practice Problems and Quality Solution

Practice 10.2 # 7, 40, 41;   10.3 # 7, 15
Quality Solution 10.3: # 2

Math 114, Taylor Series Revisited (Sections 10.2, 10.3A)

Name: ________________________________ **Section:** ________

Read Section 10.2 and the beginning up Section 10.3, up through Example 1. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Binomial Series Expansions.** Consider the function $f(x) = \frac{1}{1+x}$.

   (a) Repeatedly differentiate $f(x)$: find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Simplify along the way!

   (b) Evaluate $f(x)$ and its derivatives at $x = 0$: find $f(0)$, $f'(0)$, $\ldots$, $f^{(4)}(0)$.

   (c) Find the first five Taylor coefficients for $f(x)$ centered at $x = 0$, using the fact that $C_n = \frac{f^{(n)}(0)}{n!}$.

   (d) Write out the beginning of the Taylor series for $f(x)$, centered at $x = 0$. Include the first five terms.

(e) Compare your answer to the answer you would obtain by using the formula for the binomial series expansion in the bottom of page 549, as in Example 2.

2. **New Series by Substitution.** Reread the beginning of Section 10.3, including Example 1. Using the Taylor series for $f(x) = \frac{1}{1+x}$, find a Taylor series for $g(x) = \frac{1}{1+2x}$. For what values of x does this series converge?

3. What struck you in reading this section? What is still unclear? What questions do you have?

Main Points:

1. Integrating and differentiating Taylor series
2. Multiplying and substituting Taylor series
3. Applications of Taylor series

Overview

Taylor series give us a way to replace hard-to-work-with functions with simpler ones, namely polynomials. For example, the error function in probability is $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. By the Fundamental Theorem of Calculus, $\mathrm{erf}(x)$ is an antiderivative for e^{-x^2}. Since we know the Taylor series about $t = 0$ for e^t, we can find the Taylor series for e^{-t^2} by substitution, then if we simply treat this series like a polynomial with infinitely many terms, we can easily integrate it to find a Taylor series about $x = 0$ for $\mathrm{erf}(x)$. This means we have replaced the mysterious erf function with a family of simple approximating functions, polynomials!

Treating Taylor series as polynomials with infinitely many terms, we can differentiate them, integrate them, and multiply them to obtain new Taylor series. We can then use these Taylor series to answer questions about hard-to-work-with functions, using the approximating polynomials.

Assignments

1. Reading Assignment

Read Section 10.3. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

10.3 # 6, 25, 32, 34, 41, 46

3. Practice Problems and Quality Solution

Practice 10.3: # 8, 33, 42, 47
Quality Solution 10.3: # 40

Math 114, Using Taylor Series (Section 10.3B)

Name: ____________________ **Section:** __________

Read Section 10.3. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the following questions.

Reading Questions

1. **Differentiating a Taylor Series.** Reread Example 2. Continue this example to find a Taylor series about $x = 0$ for $\dfrac{2}{(1-x)^3}$ from the series for $\dfrac{1}{(1-x)^2}$, as outlined below.

   (a) What is the derivative of $\dfrac{1}{(1-x)^2}$? (Rewrite the function as $(1-x)^{-2}$, then use the power rule and the chain rule.)

   (b) What is the Taylor series about $x = 0$ for $\dfrac{1}{(1-x)^2}$? (This is given in Example 2.)

   (c) Differentiate the Taylor series about $x = 0$ for $\dfrac{1}{(1-x)^2}$ term by term.

   (d) What is the Taylor series about $x = 0$ for $\dfrac{2}{(1-x)^3}$? (This should be your answer in (c).)

2. **Integrating a Taylor series.** Reread Example 3. Imitate the ideas in this example to find a Taylor series about $x = 0$ for $\ln(1-x)$ from the series for $\dfrac{1}{1-x}$, as outlined below.

   (a) Find $\displaystyle\int \frac{1}{1-x}\,dx$.

   (b) What is the Taylor series about $x = 0$ for $\dfrac{1}{1-x}$? (Look back at Example 1.)

   (c) Integrate the Taylor series about $x = 0$ for $\dfrac{1}{1-x}$ term by term. (Don't forget the $+C$!)

   (d) What is the Taylor series about $x = 0$ for $\ln(1-x)$? (Use your answer from (c).)

3. What struck you in reading this section? What is still unclear? What questions do you have?