Main Points:

1. The practical meaning of the derivative
2. Using the derivative for linear approximation
3. Using the second derivative (concavity) to determine whether the linear approximation is an over-estimate or an under-estimate.

Overview and Example

Recall that the derivative is an **instantaneous rate of change**. Graphically, this instantaneous rate of change is represented as the **slope of a tangent**. We can use an instantaneous rate of change of a quantity to make predictions about how much the quantity will change in the near future. This is the idea of **linear approximation**.

For example, if I have 2 inches of flood water in my basement at 9:00 am and the rate at which flood water is rising at that moment is 4 inches per hour, then I expect to have 3 inches (3 inches = 2 inches + (4 inches per hour)·(1/4 hour)) of water in my basement at 9:15. However, it would be unreasonable to predict that I would have 98 inches of water (98 inches = 2 inches + (4 inches per hour)·(24 hours)) by 9:00 am the next day, because, while it is reasonable to assume that the rate at which water is rising is approximately constant over a short interval of time (like 15 minutes), it is unreasonable to expect the rate to be constant over a long interval of time (24 hours.)

This is the intuitive idea behind using a tangent line to approximate a function locally. If the height, in inches, of water in my basement t hours after 9:00 am is $h(t)$. Then the fact that there is two inches of water in my basement at 9:00 am means $h(0) = 2$, and the fact that the water is rising at a rate of 4 inches per hour at 9:00 am means that $h'(0) = 4$. Locally, the tangent line, represented by the linear function $L(t) = 2 + 4t$, is a reasonable approximation for $h(t)$. So the height of water at 9:15 is reasonably approximated as: $h(1/4) \approx L(1/4) = 2 + 4(1/4) = 3$ inches. However, the height of the water at 9:00 the following day, which is $h(24)$, cannot reasonably be approximated by $L(24) = 2 + 4(24) = 98$ inches.

Assignments

1. Reading Assignment

Read Section 2.4 to review the practical meaning of the derivative; you may omit Example 5. Read Section 3.9, on linear approximation, focusing on the first part, "The Tangent Line Approximation," page 169. Hopefully this material is familiar to you from Calc I.

Words and phrases in *italics* are important words and phrases. Formulas in blue boxes are important formulas. Pay attention to these things and take notes on them in your notebook!

2. Discussion Problems

2.4 #1, 2, 36,   3.9 # 1, 4, 8, 15, 21

3. Practice Problems and Quality Solution

Practice: 2.4: #5, 6,   3.9 # 2, 5, 12(a)(b), 18 Quality Solution: 3.9 #6

Main Points:

1. Extending the idea of linear approximation to quadratic approximation
2. Taylor polynomials of higher degree

Overview

Recall that if a function is concave up, then a forward linear approximation will be an under-estimate. (Similarly, if a function is concave down, then a forward linear approximation will be an under-estimate.) We can improve our estimate, taking concavity into account by adding a quadratic term whose coefficient comes from the second derivative. This is the idea behind a quadratic approximation. Extending this idea, we can make better and better estimates by constructing polynomials of higher degree whose coefficients come from higher order derivatives. Such polynomials are called **Taylor polynomials**.

Assignments

1. Reading Assignment

Read Section 10.1, focusing on pages 538-539. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the reading questions on next two pages.

2. Discussion Problems

10.1 # 3, 13, 17, 23

3. Practice Problems and Quality Solution

Practice: 10.1 # 1, 15, 18, 25-28, 29
Quality Solution: 10.1 #12

Main Points:

1. The idea of a Taylor series as a Taylor polynomial of infinite degree
2. Limitations in the scope of a Taylor series approximating a function

Overview

Recall that the quadratic approximation for a function improves the linear approximtion for the function by taking concavity into account, and this trend continues with higher degree Taylor polynomials: the higher the degree, the better the approximation, at least locally. In this section, we consider the family of all Taylor polynomials for a given function, centered at a given point, by looking at a **Taylor series**.

Assignments

1. Reading Assignment

Read Section 10.1, focusing on pages 546-547. Take notes in your notebook, making sure to include words and phrases in italics and formulas in blue boxes. Then answer the reading questions on next page.

2. Discussion Problems

10.2 # 3, 5, 16, 27(a)*
*For #27(a), you may use *Mathematica* or a graphing calculator to graph the Taylor polynomials.

3. Practice Problems and Quality Solution

Practice: 10.2 # 6, 11, 17, 25*, 31
Quality Solution: 10.2 #10
*For #25, you may use *Mathematica* or a graphing calculator to graph the Taylor polynomials.

Main Points:

1. Estimating accumulated change over a long interval
2. Estimating the area area under a curve using left and right sums
3. The definite integral as exact accumulated change or exact area

Overview

Recall that we can approximate the change in a quantity over a short interval of time using an instantaneous rate of change. (This is linear approximation.) If we want to approximate the **accumulated change** in a quantity over a *long* interval of time, we can divide the long interval into short subintervals, estimate the changes in the quantity over each subinterval, and then add up these changes to get an estimation for the accumulated change over the long time interval. (Such a sum is called a **Riemann sum**.)

Dividing up the long interval into *more* subintervals and repeating the process *improves* our estimate.

If we have a *graph* of a positive rate function $r(t)$ versus time, then the multiplication of the rate at time $t = a$ by a short time interval Δt represents the area of a rectangle of height $r(a)$ and width Δt. Thus estimating net change can be understood as estimating the **area under a curve** using rectangles, whose heights are determined by the graph of the function. Dividing the interval into more subintervals results in more rectangles; the more rectangles we use, the better our estimate will be.

For examples with distance and velocity, see Section 5.1.

We find *exact* accumulated change and *exact* areas under curves using limits. As the number of subintervals (rectangles) increases, the approximation gets better and better; the limit is called the **definite integral**.

If a rate function $r(t)$ is *negative*, this means that the quantity is *decreasing* and the net change over a time interval is negative. Because of this, when we talk about the definite integral as the “area under the curve,” we really mean that it is the **signed area between the curve and the x-axis**: the signed area is positive when the curve is above the x-axis and negative when the curve is below the x-axis.

Assignments

1. Reading Assignment.

Read 5.1 and 5.2. Take notes, and answer the reading questions.

2. Discussion Problems.

5.1 # 15(a)(b)(c), 36, 37, 5.2 # 11, 29, 41

3. Practice Problems and Quality Solution.

Practice: 5.1 # 4, 39, 5.2 # 13
QS: 5.1 #14

Main Points:

1. Midpoint Rule, Trapezoidal Rule
2. Over/under estimates, errors
3. Simpson's Rule

Overview

Recall that we can estimate the value of a definite integral using a left sum, which uses rectangles whose heights are determined by the value of the function on the *left* endpoints of the subintervals, or a right sum, which uses rectangles whose heights are determined by the value of the function the *right* endpoints.

In this section, we refine our estimates using more sophisticated numerical methods. The **midpoint rule** uses rectangles whose heights are determined by the value of the function at the *midpoints* of the subintervals. The **trapezoidal rule** takes the *average* of the left and right sums. **Simpson's rule** takes a *weighted average* of the estimates obtained from the midpoint and trapezoidal rules.

The reasons for averaging the left and right sums in the trapezoidal rule and for taking a weighted average of the midpoint and trapezoidal rule in Simpson's rule become apparent when we look at the **errors** of our estimates. For example, for an increasing function the left sum *underestimates* the integral (a positive error) whereas the right sum *overestimates* the integral (a negative error). This is the motivation for *averaging* the two (to cancel out the errors) in the trapezoidal rule.

Assignments

1. Reading Assignment

Read Section 7.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.5 # 1, 15, 17, and one additional problem; see below.

3. Practice Problems and Quality Solution

Practice: 7.5 # 4, 16, and one additional problem; see below.
Quality Solution: 7.5 #18

Additional Discussion Problem

We will approximate the value of $\int_0^1 \cos(x^2)\,dx$. Use *Mathematica* for your computations.

(a) Use *Mathematica* to graph the function: `Plot[Cos[x^2], {x, 0, 1}]`. Sketch the graph below. In particular, make sure you have the concavity correct.

(b) Approximate the integral using LEFT(4), RIGHT(4), MID(4), and TRAP(4). Use *Mathematica*, and round to six decimal places.

(c) Use your sketch of the graph in (a) to determine which of your estimates in (b) are overestimates and which are underestimates. Which is your best underestimate? Which is your best overestimate?

(d) Approximate the integral using SIMP(4).

Additional Practice Problem

In this problem we will estimate $I = \int_0^1 \sin(\frac{1}{2}x^2)\,dx$.

(a) Use *Mathematica* to graph the function: `Plot[Sin[(1/2)x^2], {x, 0, 1}]`. Sketch the graph below. In particular, make sure you have the concavity correct.

(b) List the values LEFT(n), RIGHT(n), MID(n), and TRAP(n), and I in increasing order (smallest to largest).

(c) Use *Mathematica* to compute LEFT(5), RIGHT(5), MID(5), TRAP(5), and SIMP(5).

Main Points:

1. FTC 1: Evaluating definite integrals using antiderivatives.
2. FTC 2: Constructing antiderivatives using the definite integral.

Overview

Recall that, given a function $r(t)$ for the rate at which some quantity Q is changing, we can *estimate* the net change in Q over a long time interval $[a, b]$ by dividing up the long interval into short subintervals, using linear approximation to estimate the change in Q over each subinterval, and adding up these changes. The *exact* accumulated net change is obtained by taking a limit of Riemann sums; this limit is the definite integral: $\Delta Q = Q(b) - Q(a) = \int_a^b r(t)\,dt$.

This is the idea behind the **first part of the Fundamental Theorem of Calculus**, which says that if $F(x)$ is a function with a continuous derivative $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$. Since f is a derivative of F, we say F is an antiderivative of f. This means that we can evaluate definite integrals *exactly* whenever we can find *antiderivatives*.

This same insight allows us to *construct* antiderivatives. Again, suppose $r(t)$ is a function that tells the rate at which some quantity Q is changing. Now suppose we want to know the accumulated net change in Q for many different time intervals. We fix a given intial time $t = a$ and let $A(x)$ be the net change from $t = a$ to $t = x$, for many different x-values. Graphically, this is represented by finding the (signed) area between the graph of r and the t-axis from a to x. So $A(x)$ is sometimes called the "area-so-far" function. We know that this area is represented by a definite integral: $A(x) = \int_a^x r(t)\,dt$.

Since $A(x)$ tells the net change in Q from $t = a$ to $t = x$, We can find a formula for $Q(x)$ if we know the initial quantity Q_0:

$$Q(x) \quad = \quad Q_0 \; + \int_a^x r(t)\,dt \quad = \quad Q_0 + A(x)$$

Since r is the derivative of Q, Q is an antiderivative of r. Notice that every choice of initial value Q_0 will give an antiderivative for r. In particular, choosing $Q_0 = 0$ shows that the area-so-far function itself is an antiderivative for r.

This is the idea behind the **second part of the Fundamental Theorem of Calculus**, which says that, for a continuous function $f(x)$, we can construct an antiderivative function $F(x)$ as follows: choose a number a in the domain of f and let $F(x) = \int_a^x f(t)\,dt$.

Assignments

1. Reading Assignment

Read Sections 5.3, 6.1, and 6.4. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

5.3 # 16, 19, 6.1 # 7, 6.4 # 3*, 9
*For 6.4 #3, replace the lower limit with 0.00001, as in Example 1, page 342, and use SIMP(2).

3. Practice Problems and Quality Solution

Practice: 5.3 # 21, 6.1 # 3, 10, 6.4 # 8
Quality Solution: 6.4 #22

Main Points:

1. New terms: differential equation, general solution of differential equation
2. Finding solutions to simple differential equations, verifying solutions to differential equations
3. New terms: initial value problem, initial condition, particular solution, equillibrium solution
4. Solving an initial value problem, given the general solution and an initial condition

Overview

A **differential equation** is simply an equation involving a derivative. A simple example is the equation $\frac{dy}{dx} = 2x$. Notice that this equation is true for the function $y = x^2$ and the function $y = x^2 + 5$, since these are antiderivatives of $2x$. Such functions are called **particular solutions** of the differential equation. The most general form of the solution is, of course, $y = x^2 + C$, where C is a real number. This is called the **general solution**. It represents a whole *family* of solutions. Section 6.3 discusses differential equations like this, that can be solved using antiderivatives.

Another simple differential equation is $\frac{dy}{dx} = 2y$. This equation *cannot* be solved using antiderivatives as above, but we might be able to solve it by gussing and checking. Can you think of a function whose derivative is simply 2 times itself? How about $y = e^{2x}$? We can check that this is a solution simply by differentiating: $\frac{d}{dx} e^{2x} = 2 \cdot e^{2x}$. This shows that, when $y = e^{2x}$, $\frac{dy}{dx} = 2y$, i.e. $y = e^{2x}$ is a solution to the differential equation. It turns out that every function of the form $y = Ce^{2x}$, where C is a real number, is also a solution, as we can check: $\frac{d}{dx} Ce^{2x} = C(2e^{2x}) = 2 \cdot (e^{2x})$.

We can find particular solutions from the general solution, if we also have an **initial condition**. For example, if the general solution is $y = Ce^{2x}$ and we have the initial condition $y(0) = 5$, we can find the particular solution by substituting in $x = 0$ and $y = 5$ and solving for C as follows: $5 = Ce^{2 \cdot 0} = Ce^0 = C$. Thus the particular solution is $y = 5e^{2x}$. A problem of this sort is called an **initial value problem**.

Even without a formula for the general solution of a differential equation, we can often determine quite a bit about the family of solutions: we can use qualitative analysis to sketch graphs and numerical methods to generate tables of data approximating the solution. Sections 11.1-11.3 discuss these methods in depth.

Assignments

1. Reading Assignment

Read Sections 6.3 and 11.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

6.3 # 5, 7, 12, 11.1 # 3, 9, 11, 25

3. Practice Problems and Quality Solution

Practice: 6.3 # 8, 10, 16*, 11.1 # 2, 8, 12
*For 6.3.16, your final answer will be in terms of k.
Quality Solution: 11.1 #24

Main Points:

1. Generating a slope field from a differential equation
2. Sketching solution curves using a slope field

Overview

Given a differential equation that expresses $\frac{dy}{dx}$ in terms of x and y, we can generate a table of values with x and y as inputs and $\frac{dy}{dx}$ as outputs. Thus, for each point (x, y) in the plane, we can sketch the slope, $\frac{dy}{dx}$, of the solution curve passing through that point (as long as it is defined). The collection of all these short lines is called a **slope field**, and it gives use a way to sketch solution curves for the differential equation.

Assignments

1. Reading Assignment

Read Section 11.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

11.2 # 6, 13, 17

3. Practice Problems and Quality Solution

Practice: 11.2 # 5, 14
Quality Solution: 11.2 # 18

Main Points:

1. Using separation of variables to find the general solution of certain differential equations
2. Writing a differential equation to model a real-life situation
3. Stable and unstable equillibrium solutions

Overview

A separable differential equation is one that can be written in the form: $y' = f(x)g(y)$. For example:

$$y' = xy^3 \qquad y' = x^2\, y^{-2} \qquad y' = 6x^2/(2y + \cos y)$$

are separable differential equations. To solve a separable differential equation, write the derivative in Leibnitz notation (dy/dx instead of y'), write the differential equation in "differential form," i.e. with all the x-values on one side, with dx and all the y-values on the other side, with dy, and integrate both sides.

An example of a separable differential equation that occurs in application is the equation modelling unconstrained population growth. This model operates under the assumption that a population will grow at a rate directly proportional to the size of the population, P. In other words $\frac{dP}{dt}$ is proportional to P.

$$\frac{dP}{dt} = k\,P \qquad \text{(for some } k > 0\text{)}$$

The positive constant k is the constant relative growth rate of the population.

According to Newton's Law of Cooling, the rate at which temperature of an object decreases is proportional to the temperature difference between the object and its surroundings. Thus the temperature of a cooling object can also be modelled by a differential equation involving a proportionality statement.

In general, when a quantity A is *directly proportional* to a quantity B, that means that there is a positive constant, say k, such that $A = kB$. The constant k is called the *constant of proportionality.*

An equillibrium solution of a differential equation is a constant solution. Equillibrium solutions for a differential equation of the form $y' = g(y)$ can be found by letting $y' = 0$ and solving for y, since this enables us to find y-values at which the rate at which y is changing is zero.

Assignments

1. Reading Assignment

Read Sections 11.4 and 11.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

11.4 # 3, 11, 14, 11.5 # 8, 27

3. Practice Problems and Quality Solution

Practice: 11.4 # 6, 11.5 # 18, 22
Quality Solutions: 11.4 # 16, 11.5 # 30*
*11.5.30 is a Challenge Problem. It will be graded on a more lenient scale.

Main Points:

1. Guess-and-check to "undo the chain rule"
2. Changing variables in an integral: choose inside function "w" and then $dw = w'(x)dx$
3. Two methods for using substitution in definite integrals

Overview

So far the only strategy we have for finding antiderivatives is to recognize them as derivatives of familiar functions, sometimes using algebra or trigonometry to rewrite a function first. Can you recognize $2e^{2x}$ as the derivative of a familiar function? It is the derivative of e^{2x}. The constant factor of 2 comes from the chain rule. For very simple examples, we can "undo the chain rule" in this way. (See Examples 1 and 2.)

For less simple examples, it helps to perform a change of variables. We give the "inside function" a name, say $w(x)$ and transform an integral having x as the variable of integration to an integral having w as the variable of integration. Remember that the chain rule says

$$\frac{d}{dx} F(w(x)) \;=\; F'(w(x)) \cdot w'(x)$$

Thus, if F is an antiderivative for f (i.e. $F' = f$),

$$\int f(w)\, w'(x)\, dx \;=\; \int f(w)\, dw \;=\; F(w(x)) \;+\; C$$

since $w'(x)\, dx = dw$. (See Examples 3-7.)

We can sometimes use substitution even if the integrand is not a constant multiple of something of the form $f(w(x))\, w'(x)$. In particular, as long as the integrand can be rewritten as $w'(x)$ times something entirely in terms of w, substitution is worth trying. See Examples 12-13.

Since substitution is a technique for finding antiderivatives, it is also useful for definite integrals. The trick is to be careful not to plug in x-values for w. There are two ways to do this. A two-step method requires finding the indefinite integral first; as an alternative, you can transform the limits of integration along with the whole integral. See Examples 9-11.

Assignments

1. Reading Assignment

Read Section 7.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.1 # 19, 25, 34, 38, 57, 61, 75

3. Practice Problems and Quality Solution

Practice: 7.1 # 10, 20, 29, 32, 58, 72, 77, 135
Quality Solution: 7.1 # 64

Main Points:

1. IBP as "reversing the product rule" to exchange a hard integral for an easier one
2. Two tricks: letting $v' = 1$; noticing a pattern of repeating derivatives

Overview

Integration by parts is a way to use the "reverse product rule" to exchange a hard integral for an easier one. Recall that the product rule can be written as:

$$\frac{d}{dx}\, u(x)\, v(x) \;\; = \;\; u'(x)\, v(x) \; + \; u(x)\, v'(x)$$

Restating in terms of integrals and rearranging gives:

$$\int u(x)\, v'(x)\, dx \;\; = \;\; u(x)\, v(x) \; - \; \int u'(x) v(x)\, dx$$

Using the shorthand $du = u'(x)\, dx$ and $dv = v'(x)\, dx$, we can rewrite this as:

$$\boxed{\int u\, dv \; = \; uv \; - \; \int v\, du}$$

IBP is a good strategy to try when the integrand is a product of two functions. In order for IBP to work, you need to be able to differentiate one of the functions and anti-differentiate the other. Choose u to be the function you want to differentiate and v' to be the function you want to anti-differentiate.

Sometimes IBP can be used even when the integrand does not look like a product of two functions. In particular, if we know the derivative of the integrand, we can let the whole integrand be u and we can let $v' = 1$. For example, this works for $\int \ln x\, dx$ and $\int \arcsin x\, dx$. See Example 3.

Sometimes IBP can be used even when neither part of the integrand becomes simpler when differentiated, if we can notice a pattern of repeating derivatives. See Examples 6-7.

Assignments

1. Reading Assignment

Read Section 7.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.2 # 5, 6, 9, 13, 15*, 2
*Hint: Imitate Example 6.

3. Practice Problems and Quality Solution

Practice: 7.2 # 8, 10, 20, 26, 73
Quality Solution: 7.2 # 16

Main Points:

1. Finding simple partial fractions decompositions by hand
2. Using partial fractions decompositions to simplify integration

Overview

The method of **partial fractions** is an algebraic technique that can be helpful for integration. In particular, the partial fractions decomposition is a way to *rewrite* a rational function as a *sum* of *simpler* rational functions, as long as the degree of the numerator is smaller than the degree of the denominator. (If the degree of the denominator is larger than the degree of the numerator, long division of polynomials can be used first. See Example 5.) It is a reverse process to adding rational functions, and as such requires "undoing the common denominator."

We use the partial fractions decomposition to rewrite rational integrands as sums of simpler rational functions. To evaluate these simpler integrals it may be necessary to use a substitution. Recall some basic antiderivatives:

$$\int \frac{dx}{x} = \ln|x| + C\,; \qquad \int \frac{dx}{x^p} = \frac{1}{(1-p)\,x^{p-1}} + C\,, \quad (p>1)\,; \qquad \int \frac{dx}{1+x^2} = \arctan(x) + C$$

Assignments

1. Reading Assignment

Read Section 7.4, up to but not including the part about Trigonometric Substitutions. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.4 # 15, 17, 39, 48

3. Practice Problems and Quality Solution

Practice: 7.4 # 16, 41, 49, 72
Quality Solution: 7.4 # 44

Main Points:

1. Basic trig substitution: when integrand is similar to the derivatives $\arcsin x$ or $\arctan x$.
2. More general trig substitution: when integrand involves $x^2 + a^2$ or $\sqrt{a^2 - x^2}$; using triangle.
3. Completing the square before using a trig substitution

Overview

Recall that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \;=\; \arcsin x \;+\; C \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx \;=\; \arctan x \;+\; C$$

For integrals that are very similar to these, a simple w-substitution or a basic trig substitution can be used, as in Examples 7 and 10:

Ex. 7: $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx \;=\; \arcsin(\tfrac{x}{2}) \;+\; C$ **Ex. 10:** $\displaystyle\int \frac{1}{9+x^2}\,dx \;=\; \tfrac{1}{3}\arctan(\tfrac{x}{3}) \;+\; C$

More generally, when an integral involves something of the form $\sqrt{a^2 - x^2}$ or $a^2 + x^2$, a trig substitution may be useful.

A trig substitution looks a little different from the simple w-substitutions we discussed in 7.1; instead of identifying something in the integrand as an "inside function," we let x be a trig function in terms of θ then change the variable of integration from x to θ. Then we use a trig identity to simplify the integrand before integrating. If we are evaluating an indefinite integral, it is necessary to change variables back from θ to x after integrating. Often this requires **constructing a triangle** and labeling the sides using SOH-CAH-TOA and the Pythagorean Theorem, as in Example 9.

Finally, trig substitutions can also be useful after **completing the square** to rewrite part of the integrand in the form $\sqrt{a^2 - (x-h)^2}$ or $a^2 + (x-h)^2$, as in Examples 12 and 13.

Assignments

1. Reading Assignment

Read the part about Trigonometric Substitutions in Section 7.4, starting on page 380. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.4 # 20, 21, 23, 55, 61*
*Use formula IV-20 in the Table of Integrals at the back of the textbook.

3. Practice Problems and Quality Solution

Practice: 7.4 #22, 24, 31, 56*, 71*
Quality Solution: 7.4 # 60*
*Use formulas IV-18 and IV-21, IV-22 for 56, 60, and 71 respectively.

Main Points:

1. Limit as the technical underpinning of calculus
2. Limits describing local behavior of functions
3. Limits describing end behavior of functions

Overview

The formal mathematical notion of a **limit** is the essential technical idea underlying calculus: it is necessary for careful discussions of continuous functions, derivatives, and definite integrals, which we have studied, as well as for the convergence of improper integrals, infinite sequences, and infinite series, which we have yet to study.

Limits are useful for describing **local behavior** of functions, especially where they are undefined. For example, if we notice that the y-values of $\frac{\sin x}{x}$ seem to get closer to $y = 1$ as the x-values get closer and closer to zero, we say that the limit of $\frac{\sin x}{x}$ as x approaches zero is 1. Note that this describes the behavior of the function *near but not at* the number $x = 0$. In this example, the function is undefined at $x = 0$, but this does not negate the observable trend that the y-values are approaching 1; it indicates that the graph of has a *hole* at $x = 0$. (See Example 1.)

Definition Suppose $f(x)$ is defined on some interval around c, except perhaps at the point $x = c$. Then we write $\lim_{x\to c} f(x) = L$ and say *the limit of $f(x)$, as x approaches c, equals L* if we can make the values of $f(x)$ as close to L as we like by taking x sufficiently close to c (on either side of c) but not equal to c.

Similarly we can talk about the limit from the left and the limit from the right, if we only mean to discuss x approaching c from the left, or right, respectively.

When a function increases without bound (informally: the values "go to infinity") we use the infinity symbol (∞) to denote the limit, even though the limit technically does not exist, because the values of the function do not approach a specific number.

Limits are also useful for describing the **end behavior** of functions, i.e. what happens as x becomes larger and larger (positive or negative). If a function f approaches a specific number L as x gets larger and larger (positive), we say that *the limit of $f(x)$ as x approaches infinity is L* and we write $\lim_{x\to\infty} f(x) = L$. Similarly, we write $\lim_{x\to-\infty} f(x) = L$ if f approaches L as x becomes larger and larger negative.

Assignments

1. Reading Assignment

Read Section 1.8. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

1.8 # 1, 3, 29, 51, 56, 59

3. Practice Problems and Quality Solution

Practice: 1.8 # 2, 52, 57, 58
Quality Solution: 1.8 # 30

Main Points:

1. Using l'Hopital's rule to evaluate limits of quotients
2. Using limits of quotients to describe dominance
3. Variations on l'Hopital's Rule

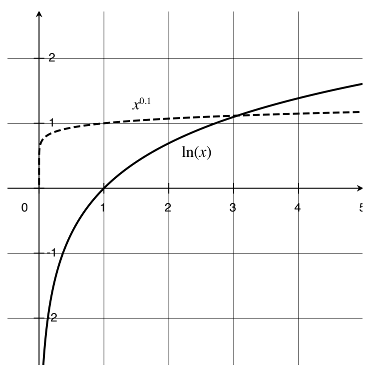
Overview

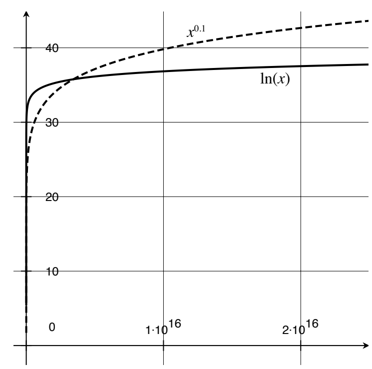
This section gives us a way to evaluate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The trick is to use **l'Hopital's rule**, which says that you can take the derivative of the top and the derivative of the bottom and *then* take the limit of *that.* Intuitively, if both the numerator and the denominator are shrinking (or growing), we use derivatives to tell us which one is shrinking (or growing) at a *faster rate.*

One application is determining the **dominance** of one function over another. For example, we can use l'Hopital's rule to prove that although $\ln x$ and x^p (for $p > 0$) both grow without bound as x increases, the power function will always surpass the logarithmic function, eventually. We do this by looking at their quotient and taking a limit as x approaches infinity.

$$\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{p\cdot x^{p-1}} = \lim_{x\to\infty} \frac{1}{p\cdot x^p} = 0$$

Note that the limit is of the form $\frac{\infty}{\infty}$, so it is valid to apply l'Hospital's rule. The $\ln x$ in the numerator is trying to make the limit go to infinity, but the x^p in the denominator is trying to make the limit go to zero; it is a competition, and l'Hopital's rule tells us who wins: since the final limit is zero the power function in the denominator wins. This means the power function dominates the logarithmic function. This is somewhat surprizing since for small p-values the power function does not appear to grow very quickly.





It is sometimes possible to use l'Hopital's Rule to evaluate limits of the form $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, or 1^∞, but it is necessary to rewrite the function as a *quotient* so that the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying l'Hopital's rule. See Examples 6, 7, and 8.

Assignments

1. Reading Assignment

Read Section 4.7. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.7 # 1, 5, 6, 14, 34, 45

3. Practice Problems and Quality Solution

Practice: 4.7 # 4, 7, 15, 46, 47
Quality Solution: 4.7 # 44

Main Points:

1. Two kinds of improper integrals
2. Using limits to describe improper integrals

Overview

Our original discussion of the definite integral does not allow for integrating over an infinite interval or integrating functions that are unbounded at a point, but such integrals, called **improper integrals** do arise in applications. We use limits to describe such integrals: some of which have a finite value, some of which do not. An improper integral with a finite value is called **convergent**, whereas an improper integral without a finite value is called **divergent**.

Assignments

1. Reading Assignment

Read Section 7.6. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

7.6 # 3, 5, 7, 23, 24, 43

3. Practice Problems and Quality Solution

Practice: 7.6 # 4, 6, 9, 13, 19, 20
Quality Solution: 7.6 # 36

Main Points:

1. Representing motion in the plane with parametric equations
2. Finding speed and velocity

Overview

Recall that motion along a straight line can be described by a position function $s(t)$, and its derivative is the velocity function $v(t) = \frac{ds}{dt}$. Velocity can be positive or negative; the sign indicates the direction of motion. Speed, on the other hand, is always positive, and it is given by the magnitude (absolute value) of the velocity.

Motion in the plane can be described by a pair of functions: $x(t)$ and $y(t)$, representing the x-coordinate and y-coordinate of the position at time t, respectively. The equations for x and y in terms of t are called **parametric equations** because they express x and y in terms of a common **parameter**, namely t.

The **velocity** of an object moving in the plane is also represented by a pair of functions: the velocity in the x-direction is $v_x(t) = \frac{dx}{dt}$ and the velocity in the y-direction is $v_y(t) = \frac{dy}{dt}$. The velocity vector is a way of expressing both of these velocities simultaneously. **Speed** simply describes how fast an object moves along the direction of its motion; it is the magnitude (length) of the velocity vector.

Assignments

1. Reading Assignment

Read the first part of Section 4.8, up to and including Example 8 on page 253. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.8 # 1, 7, 9, 21, 22, 31, 46

3. Practice Problems and Quality Solution

Practice: 4.8 # 3, 5, 23, 30, 39, 49
Quality Solution: 4.8 # 26

Main Points:

1. Representing a curve with parametric equations
2. Eliminating the parameter to find a Cartesian equation for a curve given by parametric equations
3. Tangent lines, slope and concavity of parametric curves
4. Area under parametric curve

Overview

Recall that the motion of a particle in the xy-plane can be described using parametric equations, which describe the x-coordinate and the y-coordinate of the particle at a given time t. The path in the xy-plane traced out by the particle over time is an example of a **parametric curve**, a curve whose coordinates are given by equations expressed in terms of a common variable called the **parameter**. The parameter is usually denoted t, suggesting time, but it is legitimate to use any variable for the parameter, and, in applications, the parameter does not necessarily represent time. Sometimes it is possible to **eliminate the parameter** to obtain an equation for the curve involving only x and y, as in Example 1, when the parametric curve given by $x = \cos t$, $y = \sin t$ was rewritten as $x^2 + y^2 = 1$.

For a curve in the xy-plane, the **slope of a tangent line** (assuming there is a well-defined tangent line!) to the curve at the point (a, b) is $\frac{dy}{dx}|_{(a,b)}$. We use the Chain Rule to find $\frac{dy}{dx}$ for a curve in parametric form.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad (\text{if } x'(t) \neq 0)$$

Note that this will give us a formula for $\frac{dy}{dx}$ in terms of t. Let $m(t)$ be $\frac{dy}{dx}$ as a function of t. Then, to determine **concavity**, we need $\frac{d^2y}{dx^2}$, which is the derivative of m with respect to x:

$$\frac{d^2y}{dx^2} = \frac{dm}{dx} = \frac{m'(t)}{x'(t)} \quad (\text{if } x'(t) \neq 0)$$

Since $dx = x'(t)dt$, the **signed area** between a parametric curve and the x-axis from $x(\alpha)$ to $x(\beta)$ is:

$$\int_{x(\alpha)}^{x(\beta)} y\, dx = \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt$$

as long as the curve is traversed exactly once, from left to right, as t increases from α to β.

Assignments

1. Reading Assignment

Read the second part of Section 4.8, **as well as the hand-out on finding area under a parametric curve**. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

4.8 # 27, 45, 47, 52, and one additional problem (see below)

3. Practice Problems and Quality Solution

Practice: 4.8 # 29, 51, 54, and one additional problem (see below)
Quality Solution: 4.8 # 28

Additional Discussion Problem.

Sketch the curve $x = 3\cos\theta$, $y = 4\sin\theta$, $0 \leq \theta \leq 2\pi$, and find the area that it encloses. (Hint: Use symmetry. Divide the area into four equal pieces and find the area of one piece, then multiply by four to get the total area.)

Additional Practice Problem.

Sketch the curve $x = 9 + e^t$, $y = t - t^2$, and find the area enclosed by the curve and the x-axis.

Areas Enclosed by Parametric Curves

Suppose we have a parametric curve lying above the x-axis and the curve is traversed exactly once, from left to right, as the parameter moves from $t = \alpha$ to $t = \beta$. The vertical distance from the curve to the x-axis is given by $y(t)$ and differential is $dx = x'(t)dt$. Thus the area between the curve and the x-axis, from $x(\alpha)$ to $x(\beta)$ is:

$$A \;=\; \int_{x(\alpha)}^{x(\beta)} y\,dx \;=\; \int_{\alpha}^{\beta} y(t)\,x'(t)\,dt$$

Example 1. Find the area between the parametric curve $x = t - 1$, $y = t^2 + t$ and the x-axis from $x = 0$ to $x = 1$.

Notice that as t increases, $x = t - 1$ also increases, so as "time" moves forward, the curve is traversed from left to right. The parameter values corresponding to $x = 0$ and $x = 1$ are $t = 1$ and $t = 2$, respectively. Notice that y is positive when $1 \le t \le 2$. Thus the area between the curve and the x-axis from $x = 0$ to $x = 1$ is:

$$A \;=\; \int_1^2 y(t)x'(t)\,dt \;=\; \int_1^2 (t^2+t)(1)dt \;=\; \int_1^2 t^2 + t\,dt \;=\; \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2\right)\Big|_1^3 \;=\; \tfrac{23}{6}$$

To check our work, we can eliminate the parameter and integrate with respect to x. Since $x = t - 1$, $t = x + 1$, and $y = (x+1)^2 + (x+1) = x^2 + 3x + 2$. Thus the area is

$$A \;=\; \int_0^1 x^2 + 3x + 2\,dx \;=\; \left(\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x\right)\Big|_0^1 \;=\; \tfrac{23}{6}$$

Example 2. Find the area enclosed by the parametric curve $x = e^t$, $y = 1 - t^2$ and the x-axis.

Notice that as t increases, $x = e^t$ also increases, so as "time" moves forward, the curve is traversed from left to right. Also notice that the x-intercepts of the curve will occur when $y = 0$, namely when $t = -1$ and $t = 1$, and when t is between -1 and 1, y is positive. Thus the area enclosed by the curve and the x-axis is given by the integral

$$A \;=\; \int_{-1}^{1} y(t)\,x'(t)\,dt \;=\; \int_{-1}^{1} (1-t^2)e^t\,dt$$

We can evaluate this integral using IBP. Let $u = 1 - t^2$ and $dv = e^t dt$. Then $du = -2tdt$ and $v = e^t$. Thus:

$$\int (1-t^2)e^t\,dt \;=\; (1-t^2)e^t \;-\; \int (e^t)(-2t)\,dt \;=\; (1-t^2)e^t \;+\; 2\int te^t\,dt$$

We need to use IBP again: with $u = t$, $dv = e^t dt$, we get $du = dt$ and $v = e^t$, so

$$\int te^t\,dt \;=\; te^t \;-\; \int e^t\,dt \;=\; te^t - e^t + C \;=\; (t-1)e^t + C$$

Substituting back into the original integral, we get

$$\int (1-t^2)e^t\,dt \;=\; (1-t^2)e^t \;+\; 2\int te^t\,dt \;=\; (1-t^2)e^t \;+\; 2(t-1)e^t + C \;=\; -(1-2t+t^2)e^t + C$$

So we can conclude

$$A = \int_{-1}^{1} (1-t^2)e^t\,dt \;=\; -(1-2t+t^2)e^t\Big|_{-1}^{1} \;=\; 0 + 4e^{-1} \;=\; 4/e$$

Main Points:

1. Area between two curves
2. Average value of a continuous function on an interval

Overview

Area Recall that we use a definite integral to find the (signed) area between a curve and the x-axis. If $f(x) \geq 0$ on an interval $[a, b]$, then the definite integral gives a literal area:

$$(\text{area between } f(x) \text{ and } x\text{-axis from } x = a \text{ to } x = b) \quad = \quad \int_a^b f(x)\,dx$$

Similarly, if a function $f(x) \geq g(x)$ on an interval $[a, b]$, the **area between the two curves from** $x = a$ **to** $x = b$ is obtained by subtracting the smaller area from the greater area:

$$(\text{area between } f(x) \text{ and } g(x) \text{ from } x = a \text{ to } x = b) \quad = \quad \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \quad = \quad \int_a^b \big(f(x) - g(x)\big)\,dx$$

When two continuous curves cross each other more than once, we can find the area of the region (or regions) between the two curves using this idea, once we have determined (a) the intersection points of the two curves and (b) which curve lies above the other in each region.

Average Value Another application of the integral is finding the average value of a quantity that changes in a continuous way. For example, to estimate the average temperature in a given city over the course of a year, we might take the average of the temperatures recorded at noon on the first day of each month: add up these temperatures and divide by twelve. We could improve our estimate by using weekly, daily, even hourly temperature recordings. The sums that are used to compute these averages are Riemann sums; taking a limit as the number of recordings goes to infinity gives a definite integral. See the discussion on page 304 for the details of the derivation of the average value formula:

$$(\text{average value of } f(x) \text{ on } [a, b]) \quad = \quad \frac{1}{b-a} \int_a^b f(x)\,dx$$

Assignments

1. Reading Assignment

Read "Area Between Curves" and "The Definite Integral as an Average" in Section 5.4, pages 301-302 and pages 304-305. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

5.4 # 7, 13, 18, 35*
*Use this fact: Since $a^t = e^{\ln(a^t)} = e^{(\ln a)t}$, $P = 112(1.011)^t = 112e^{(\ln(1.011))t}$.

3. Practice Problems and Quality Solution

Practice: 5.4 # 8, 15, 20*, 25
Quality Solution: 5.4 # 34
*Do not try to find the intersection point exactly; estimate it by zooming in on the graphs.

Main Points:

1. Approximating areas and volumes with Riemann sums of "slices"
2. Calculating area and volume exactly using definite integrals

Overview

Recall that the definite integral can be used to find the area under a curve. The area is first approximated by a Riemann sum, which is the sum of the areas of rectangles, whose heights are given by the y-values of the curve and whose widths are Δx, small changes in x. Taking a limit as the number of rectangles goes to infinity gives the exact area.

We generalize this procedure to find **areas of various regions** in the plane: we slice the region into thin strips, approximate each strip by a rectangle, add up the areas of the rectangles, and take a limit as the number of rectangles goes to infintity. If we can represent this sum as a Riemann sum, then the limit is a definite integral, and we can try to find the area using an antiderivative.

In particular, to find the **area of a region enclosed by two curves** in the plane, we usually slice the region vertically (in which case the thickness of the strips is Δx) or horizontally (in which case the thickness of each strip is Δy). When slicing vertically, the heights of the approximating rectangles will be given by the vertical distance between the two curves, which can be computed by subtracting the height of the bottom curve from the height of the top curve: $h(x) = \text{Top}(x) - \text{Bottom}(x)$. When slicing horizontally, the widths of the approximating rectangles wil be given by the horizontal distance betwen the two curves, which can be computed by subtracting the left curve from the right curve: $w(y) = \text{Right}(y) - \text{Left}(y)$.

To find the **volume of a solid**, we slice the solid into thin slices, whose crossectional area is known from geometry (for example, the area of a circle, rectangle, or triangle.) We then approximate the volume of each slice by multiplying the crossectional area by the thickness, add up these volumes, and take a limit as the number of slices goes to infinity. Again, if we can represent the sum as a Riemann sum, then the limit is a definite integral and we can try to find the volume using an antiderivative.

Assignments

1. Reading Assignment

Read Section 8.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.1 # 7, 11, 12, 13, 24, 36

3. Practice Problems and Quality Solution

Practice: 8.1 # 2, 4, 18, 25, 27, 35
Quality Solution: 8.1 # 26

Main Points:

1. Finding volumes of revolution by slicing solid into disks or washers
2. Volumes of solids constructed by standing squares, semicircles, or triangles on edge in a planar region.

Overview

Recall our strategy for finding the volume of a solid: we slice the solid into thin slices, whose crossectional area is known from geometry, approximate the volume of each slice, add up these volumes, and take a limit as the number of slices goes to infinity. If we can represent the sum as a Riemann sum, then the limit is a definite integral and we can try to find the volume using an antiderivative.

We now focus our attention on finding volumes of solids constructed from planar regions in a couple specific ways.

Solids of revolution are constructed by rotating a planar region about some axis. When we slice a solid of revolution perpendicularly to the axis of rotation, each slice can be approximated by a circular disk or washer (a circular disk with a hole in the center), whose volume is easy to compute. See Examples 1-3.

Given a planar region, we may also construct a solid by **standing squares, semicircles, or triangles on edge** in this region. See Example 4.

Assignments

1. Reading Assignment

Read Section 8.2, pages 422-425, up to but not including the part about arc length. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.2 # 7, 13, 40-44

3. Practice Problems and Quality Solution

Practice: 8.2 # 8, 11, 46-50, 55
Quality Solution: 8.2 # 14

Main Points:

1. Arc length formulas
2. Using *Mathematica* for numerical integration

Overview

We can use a definite integral to express the length of a curve: we divide the curve into short segments, approximate the length of each segment by assuming the segment is straight, add up these approximate lengths to obtain a Riemann sum, and then take a limit as the number of segments approaches infinity.

As usual, *if* it is possible to find an antiderivative for the integrand, we can use the Fundamental Theorem of Calculus to evaluate the integral and find the arc length of the curve exactly. However, it turns out that, in arc length problems, frequently the integrand does *not* have an elementary antiderivative; numerical methods are needed to approximate the integral. We can use *Mathematica* to plot the curves and to estimate their arc lengths numerically.

The curve $y = x^3$ from $x = 0$ to $x = 5$ in Example 5 can be plotted with the *Mathematica* `Plot` command:

```
Plot[x^3, {x, 0, 5}]
```

The integral for the arc length is $\int_0^5 \sqrt{1 + (3x^2)^2}\, dx$. It can be estimated numerically in *Mathematica*:

```
NIntegrate[Sqrt[1+(3x^2)^2], {x, 0, 5}]
```

The parametric curve $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ in Example 6 can be plotted with `ParametricPlot`:

```
ParametricPlot[{2*Cos[t], Sin[t]}, {t, 0, 2*Pi}]
```

The integral for its arc length is $\int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\, dt$ and can be estimated numerically by:

```
NIntegrate[Sqrt[4(Sin[t])^2 + (Cos[t])^2], {t, 0, 2*Pi}]
```

Assignments

1. Reading Assignment

Read Section 8.2, pages 425-427. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

8.2 # 15*, 17*, 19, 21, 23*, 67**
*Use *Mathematica* to plot the curve and to estimate the arc length using numerical integration. The natural log is `Log`.
**Hints: Show and use the fact that $\sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2} = \frac{1}{2}(e^x + e^{-x})$. Also, at the end of the problem, you need to solve $e^b - e^{-b} = 10$ for b. Use a graph to estimate b instead of trying to find b exactly.

3. Practice Problems and Quality Solution

Practice: 8.2 # 16*,18, 22, 52
Quality Solution: 8.2 # 24*
*Use *Mathematica* to plot the curve and estimate the arc length using numerical integration. Absolute value is `Abs`.

Main Points:

1. Mathematical meaning of sequence
2. Recursive sequences
3. Convergence of sequences
4. Monotone bounded sequences

Overview

A **sequence** is an infinite list of numbers in a definite order. The numbers in the sequence are called **terms**. One way of describing a sequence is by listing the first several terms, as in these examples:

$1, 2, 3, 4, 5, 6, \ldots \qquad 1, -1, 1, -1, 1, -1, \ldots \qquad 1,\ 1/2,\ 1/4,\ 1/8, \ldots$

Another way of describing a sequence is by giving a formula for the n^{th} term of the sequence. For example the three sequences above could be represented with the following three formulas:

$a_n \ = \ n\ ,\ \ n \geq 1 \qquad b_n \ = \ (-1)^n\ ,\ \ n \geq 0 \qquad c_n \ = \ 1/2^n\ ,\ \ n \geq 0$

Some sequences are more easily described **recursively**. In a recursively defined sequence, the first term (or the first few terms) are given along with a formula for how to find successive terms. For example, the first sequence above could be defined recursively as: $a_1 = 1$, $a_n = a_{n-1} + 1$ for $n > 1$.

If the numbers a_n approach a specific, finite number L as $n \to \infty$, then the sequence is said to **converge**, and L is called the **limit** of the sequence. If a sequence does not have a limit, it is said to **diverge**.

Assignments

1. Reading Assignment

Read Section 9.1. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.1B # 13, 14, 17, 18, 21, 24, 63, 64

3. Practice Problems and Quality Solution

Practice 9.1A: # 4, 8, 11, 28, 29, 31, 42, 43, 45, 55
Quality Solution 9.1A: # 56

Practice 9.1B:# 16, 19, 20, 23, 65, 66
Quality Solution 9.1B: # 22

Main Points:

1. Mathematical meaning of series
2. Finite and infinite geometric series

Overview

Adding the terms in a sequence results in a **series**. Of course, one ought to be suspicious of whether adding infinitely many numbers can actually result in a finite number. Sometimes it does; sometimes it doesn't. A **convergent** series is one that does have a finite sum; a **divergent** series does not have a finite sum. We will use limits to make the notion of convergence precise, as we did in our discussion of improper integrals.

We may try to approximate the sum (if it exists) of an infinite series $a_0 + a_1 + a_2 + \dots$ by adding up a large (but finite) number of terms. Adding up the first n terms results in the nth **partial sum**: $S_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}$. In hopes of obtaining better and better approximations, we add more and more terms (i.e. we let n approach infinity). However, the partial sums may grow unboundedly (or exhibit other wild behavior); in such cases the series diverges. If, in contrast, the partial sums approach a specific finite number, as we add more and more terms, the series converges and the sum S of the series is defined to be the limit of the partial sums: $S = \lim_{n\to\infty} S_n$.

One of the simplest kinds of series is a **geometric series**, one in which the ratio of successive terms is a constant, called the **common ratio**. For example, $3 + 6 + 12 + 24 + 48 + \dots$ is a geometric series with common ratio 2, and $3 + 3/2 + 3/4 + 3/8 + 3/16 + \dots$ is geometric with common ratio 1/2. It is clear that the first geometric series diverges; that the second series converges is perhaps not quite as obvious, but still relatively straightforward. Imagine that there are six cupcakes left after your birthday party, so you eat three yourself, then give half of what's left to your friend (3/2 cupcakes). Your friend eats half of what you gave her and gives the rest (3/4 a cupcake) to you, who again eat half and give the rest to her (3/8 a cupcake). If you continue in this way indefinitely, how much will the two of you eat? Well, certainly not an infinite amount, because you only started with six cupcakes!

It turns out that a geometric series diverges when the absolute value of the common ratio is greater than or equal to one (again, this is fairly obvious) and converges when the absolute value of the common ratio is less than one. This can be proven rigorously, by looking at the limit of partial sums.

Assignments

1. Reading Assignment

Read Section 9.2. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.2 # 1, 2, 8, 9, 19, 23, 25, 33, 35

3. Practice Problems and Quality Solution

Practice 9.2: # 3, 4, 6, 10, 11, 21, 34, 40
Quality Solution 9.2: # 24

Main Points:

1. Convergence properties of series
2. Comparison with improper integrals
3. The harmonic series and p-series

Overview

In the previous section, we looked at a special family of infinite series: infinite geometric series; we used the formula for the sum of a finite geometric series to find the sum of a convergent infinite geometric series, using the limit of partial sums. We now expand our view to look at convergence and divergence of series more generally.

Recall that an infinite series $a_0 + a_1 + a_2 + \dots$ is convergent if the limit of partial sums is finite. In this case the sum of the series S is the limit of partial sums: $S = \lim_{n \to \infty} S_n$, where $S_n = a_0 + a_1 + \cdots + a_{n-1}$. A series is divergent if it does not converge.

Theorem 9.2 lists several straightforward **convergence properties** of series. Make sure to include the full statement of this theorem in your notes.

We next use what we know about improper integrals to discuss the convergence and divergence of infinite series. The Riemann sum of an improper integral is an infinite series; the convergence or divergence of the integral can help us determine the convergence or divergence of the series. This idea is made precise in **the integral test**. The integral test allows us to determine convergence/divergence in an important family of examples: the p**-series**, which includes as a special case the **harmonic series.**

Assignments

1. Reading Assignment

Read Section 9.3. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.3 # 1, 4, 23, 25, 33, 37
Hints: For 23, 25, and 33, use Theorem 9.2. For 33, also use the fact that $\ln(2^n) = \ln(2) \cdot n$.

3. Practice Problems and Quality Solution

Practice 9.3: # 2, 5, 6, 24, 34, 35
Quality Solution 9.3: # 38, 8*

Main Points:

1. Testing series for convergence
2. The Ratio Test

Overview

Recall that the sum of a convergent infinite series is the limit of its partial sums; if the limit of partial sums does not exist, the series is divergent and does not have a finite sum. In many cases, finding a fomula for the nth partial sum is impractical, but, fortunately, there are ways to determine whether or not a series converges without having to compute the limit of partial sums explicitly. An example we have discussed is the Integral Test; several other tests are discussed in Section 9.4. We will focus on the **Ratio Test**.

Recall that a geometric series converges if the ratio of successive terms (which is a constant, called the common ratio) has absolute value less than one. For a series $\sum a_n$ that is not geometric, the ratio a_{n+1}/a_n of successive terms will not be a constant, but if the absolute value of the ratio *approaches* a constant *less than one* as n increases, the series converges. This is the idea behind the Ratio Test.

Assignments

1. Reading Assignment

Read the part of Section 9.4 about the Ratio Test, pages 515-516. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.4 # 15, 19, 20, 51, 60

3. Practice Problems and Quality Solution

Practice 9.4: # 14, 17, 18, 52, 61
Quality Solution 9.4: # 16

Main Points:

1. Power series as "polynomials with infinitely many terms"
2. Domain of power series, radius of convergence
3. Using geometric series to find rational function for power series

Overview

Not all functions that turn out to be interesting or useful for applications can be described in terms of familiar functions. For example, the function e^{-x^2} is used in probability, but its antiderivative is not elementary: it cannot be expressed in terms of familiar functions. It is called the "error function," sometimes denoted erf. Another example is the "Bessel functions," which are used to model electromagnetic waves, heat conduction, and vibrating membranes.

One way to represent such functions is as **power series**, which can be thought of as polynomials with infinitely many terms. (Of course, there is the question of convergence.) In general, a power series about $x = a$ is of the form:

$$C_0 \;+\; C_1(x-a) \;+\; C_2(x-a)^2 \;+\; C_3(x-a)^3 \;+\; \ldots$$

The **domain** of a power series consists of all real numbers x for which the series converges. In general, the domain of a power series will be an interval centered around $x = a$. The distance from zero to either endpoint is called the **radius of convergence**. The endpoints of the interval may or may not be included. We typically use the Ratio Test to determine the radius of convergence. Usually, further work is needed to determine convergence at the endpoints.

Two extreme cases are worth discussing separately: (1) if the power series converges only at $x = a$, the domain is simply $\{a\}$ and the radius of convergence is $R = 0$ and (2) if the power series converges for all x, the domain is $(-\infty, \infty)$ and we say that the radius of convergence is $R = \infty$.

The geometric series $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and diverges otherwise. Viewed as a power series, it is a function with domain $(-1, 1)$. We can find a rational function that agrees with the power series on its domain using the formula for the sum of a geometric series:

$$\frac{1}{1-x} \;=\; \sum_{n=0}^{\infty} x^n \;=\; 1 \;+\; x \;+\; x^2 \;+\; x^3 \;+\; \ldots$$

We can use this fact to find power series representations of some functions that are similar to this one. For example, with first term 3 and common ratio $4x$,

$$\frac{3}{1-(4x)} \;=\; \sum_{n=0}^{\infty} 3\cdot(4x)^n \;=\; \sum_{n=0}^{\infty} (3\cdot 4^n)\, x^n \;=\; 3 \;+\; 12x \;+\; 48x^3 \;+\; \ldots$$

This converges when $|4x| < 1$, or $|x| < \frac{1}{4}$. Thus the domain of the power series is $(-\frac{1}{4}, \frac{1}{4})$.

Assignments

1. Reading Assignment

Read Section 9.5. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

9.5 # 5, 7, 11, 17, 18, 19, 35, 46

3. Practice Problems and Quality Solution

Practice 9.5: # 12, 13, 24, 36, 45
Quality Solution 9.5: # 16

Main Points:

1. Recall the notion of a Taylor series
2. Binomial series expansion
3. Finding new Taylor series by substitution

Overview

Recall our study of **Taylor polynomials** at the beginning of the semester. The first three Taylor polynomials for a function $f(x)$ centered at a given point $x = a$ are simply the constant approximation: $T_0(x) = f(a)$, the linear approximation $T_1(x) = f(a) + f'(a)(x - a)$, and the quadratic approximation $T_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$. In general the Taylor coefficient C_n for the $(x - a)^n$ term in the Taylor polynomial is $C_n = f^{(n)}(a)/n!$, where $f^{(n)}$ refers to the nth derivative of f, so

$$T_n(x) \;=\; f(a) \;+\; f'(a)(x-a) \;+\; \frac{f''(a)}{2!}(x-a)^2 \;+\; \frac{f'''(a)}{3!}(x-a)^3 \;+\; \ldots \;+\; \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Each successive Taylor polynomial gives a better approximation for $f(x)$ near $x = a$. We can describe the family of *all* Taylor polynomials for a given function centered at a given point using **Taylor series**.

Now that we have studied infinite series and power series in particular, we are in a position to discuss Taylor series in more depth and with more rigor. In particular, now that we have a precise understanding of convergence, we can find the radius of convergence of a Taylor series analytically (using the Ratio Test).

We take this as an opportunity to review the Taylor series for e^x, $\sin x$, and $\cos x$ and to discuss an important family of examples from Section 10.2 that we did not discuss at the beginning of the semester: the family of **binomial series** expansions, namely Taylor series for functions of the form $f(x) = (1 + x)^p$, for some real number p. In the first part of Section 10.3, we discuss how to find **new Taylor series by substitution**.

Assignments

1. Reading Assignment

Read Section 10.2 and the beginning up Section 10.3, up through Example 1 on page 553. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

10.2 # 1, 4, 27(b)*, 47; 10.3 # 1, 4, 48
*Before doing #27(b), look back at your answers for #5 and #27(a) from the beginning of the semester.

3. Practice Problems and Quality Solution

Practice 10.2 # 7, 28, 29; 10.3 # 7, 13
Quality Solution 10.3: # 2

Main Points:

1. Integrating and differentiating Taylor series
2. Multiplying and substituting Taylor series
3. Applications of Taylor series

Overview

Taylor series give us a way to replace hard-to-work-with functions with simpler ones, namely polynomials. For example, the error function in probability is $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$. By the Fundamental Theorem of Calculus, $\mathrm{erf}(x)$ is an antiderivative for e^{-x^2}. Since we know the Taylor series about $t = 0$ for e^t, we can find the Taylor series for e^{-t^2} by substitution, then if we simply treat this series like a polynomial with infinitely many terms, we can easily integrate it to find a Taylor series about $x = 0$ for $\mathrm{erf}(x)$. This means we have replaced the mysterious erf function with a family of simple approximating functions, polynomials!

Treating Taylor series as polynomials with infinitely many terms, we can differentiate them, integrate them, and multiply them to obtain new Taylor series. We can then use these Taylor series to answer questions about hard-to-work-with functions, using the approximating polynomials.

Assignments

1. Reading Assignment

Read Section 10.3. Take notes in your notebook, and answer the reading questions.

2. Discussion Problems

10.3 # 6, 21, 24, 26, 33, 40

3. Practice Problems and Quality Solution

Practice 10.3: # 8, 25, 34, 41
Quality Solution 10.3: # 32