Name: ______________________________

Names of collaborators: ______________________________

Main Points:

1. Beyond linear approximation
2. Finding Taylor polynomials
3. Radius of convergence

1. Beyond Linear Approximation

Recall from Calc 1 that if $f(x)$ is a function that is differentiable at a point $x = a$, we may approximate the values of $f(x)$ for x near a using the linear approximation:

$$f(x) \;\approx\; f(a) \,+\, f'(a)(x-a) \quad \text{(for } x \text{ near } a\text{)}.$$

Essentially, we are approximating the function $f(x)$ with the linear function $L(x) = f(a) + f'(a)(x-a)$. From a graphical perspective, we are approximating the curve $y = f(x)$ with the straight line $y = L(x)$; this is the tangent line to the curve at the point $(a, f(a))$.

One way to improve the approximation is to take the concavity of $f(x)$ at $x = a$ into account. We find a quadratic function that has the same value at $x = a$, the same derivative value at $x = a$ (so the graphs have the same slope), and the same second derivative value at $x = a$ (so the graphs have the same concavity), and use that quadratic function to approximate $f(x)$ for x near a:

$$f(x) \;\approx\; f(a) \,+\, f'(a)(x-a) \,+\, \tfrac{1}{2}f''(a)(x-a)^2 \quad \text{(for } x \text{ near } a\text{)}.$$

Continuing in this way, we have a sequence of approximations that (ideally) continue to improve, at least for x close enough to a.

Exercises.

1. Estimate $\cos(0.01)$ using a quadratic approximation for cosine. (Let $f(x) = \cos(x)$ and $a = 0$. Find a quadratic approximation for f near a, using the formula $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$, and then plug in $x = 0.01$.)

2. Taylor Polynomials

For a function $f(x)$ that is infinitely differentiable at $x = a$, we can define the nth degree Taylor polynomial for f at a to be:

$$T_n(x) \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n\,, \quad \text{where } c_k \;=\; \tfrac{1}{k!} f^{(k)}(a) \;\text{ for } 0 \le k \le n\,.$$

The notation $f^{(k)}$ is used to denote the kth derivative of f, and $k!$ is the factorial of k, which is defined to be $k! = 1 \cdot 2 \cdot \cdots \cdot (k-1) \cdot k$, for integers $k \geq 1$. In the special case that $k = 0$, we define $k! = 0! = 1$.

Notice that the first degree Taylor polynomial is simply the linearization:

$$T_1(x) \;=\; L(x) \;=\; f(a) + f'(a)(x-a)\,,$$

and the second degree Taylor polynomial $T_2(x)$ is the basis for the quadratic approximation discussed above:

$$T_2(x) \;=\; f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2\,.$$

Also notice that the "zeroth" degree Taylor polynomial is just the constant polynomial $T_0(x) = f(a)$; it could be used for the very simple approximation: $f(x) \approx f(a)$ for x near a.

In the special case that $a = 0$, sometimes the Taylor polynomials are called "Maclaurin" polynomials.

Exercises

2. Find Taylor polynomials (also called Maclaurin polynomials) for $\cos x$ at $x = 0$. Note that several of the coefficients (the c_k's) will be zero, but not all. Go up to degree four.

3. Find Taylor polynomials for $\sqrt{x}$ at $x = 1$, starting with degree 0 and going up to degree 3.

3. Radius of Convergence

Given a function $f(x)$ that is infinitely differentiable at a point $x = a$, we may graph $f(x)$ with its Taylor polynomials at $x = a$. We notice that often (but not always), the graphs of the Taylor polynomials seem to approximate the graph of $f(x)$ better and better, at least in some symmetric interval of x-values centered at $x = a$. More precisely, it seems that there is a number R such that, when x is between $a - R$ and $a + R$, $T_n(x)$ approaches $f(x)$ as $n \to \infty$. When such a number R exists, it is called the "radius of convergence."

Exercise

4. (a) Use Taylor polynomials for to find, by hand, a rational approximation to $\sqrt{3/2}$ that is accurate to at least 1/100. (You may "cheat" and use a calculator to find the errors: $T_n(3/2) - \sqrt{3/2}$.)

   (b) Could the **same** Taylor polynomials also be used to approximate $\sqrt{5}$, accurate to 1/100? (Hint: graph the Taylor polynomials and estimate the radius of convergence.)