Name: ____

Names of collaborators: ____

Main Points:

1. Methods of numerical integration (left/right/trapezoidal/midpoint)
2. Interpreting results (over/under estimates, relative accuracy of different methods)
3. Formulas for bounds for errors
4. Simpson's Rule and bounds for its error

1. Methods of numerical integration

Exercises.

1. Read the first three paragraphs of Section 7.7. What are the two situations in which it is impossible to find the exact value of a definite integral?

2. We will approximate the value of $\int_0^1 \cos(x^2)\,dx$. Use *Mathematica* (or other technology) for your computations. (See Example 1.)

   (a) You may use *Mathematica* to graph the function: `Plot[Cos[x^2], {x, 0, 1}]`, or you may use other technology. Sketch the graph below. In particular, make sure you have the concavity correct.

   (b) Approximate the integral using (i) left endpoint approximation (L_4), (ii) right endpoint approximation (R_4), (iii) the trapezoidal approximation (T_4), (iv) the midpoint approximation (M_4). Round to six decimal places.

(c) Use your sketch of the graph in (a) to determine which of your estimates in (b) are overestimates and which are underestimates.

(d) Use your best underestimate (call it m) and your best overestimate (call it M) to write an inequality bounding the integral above and below:

$$m \ < \ \int_0^1 \cos(x^2)\,dx \ < M$$

What is the range between your upper bound and your lower bound? What does this tell you about the accuracy of your approximations?

3. In this problem we will estimate $I = \int_0^1 \sin(\frac{1}{2}x^2)\,dx$.

(a) Use technology to graph the function; in *Mathematica*: `Plot[Sin[(1/2)x^2], {x, 0, 1}]`. Sketch the graph below. In particular, make sure you have the concavity correct.

(b) Use the graph to decide whether L_2, R_2, T_2, and M_2 underestimate or overestimate the integral I. (You can do this without computing L_2, R_2, T_2, and M_2.)

(c) List the numbers L_n, R_n, T_n, M_n, and I in increasing order.

(d) Use technology to compute L_5, R_5, T_5, and M_5. From the graph, which do you think gives the best estimate of I?

(e) Use your best underestimate (call it m) and your best overestimate (call it M) to write an inequality bounding the integral above and below:

$$m \ < \ \int_0^1 \sin(\tfrac{1}{2}x^2)\,dx \ < M$$

What is the range between your upper bound and your lower bound? What does this tell you about the accuracy of your approximations?

2. Formulas for bounds for error

An approximation does not have much meaning if it is not accompanied by a statement regarding the accuracy of the approximation. The statement that this piece of paper is approximately 7 inches wide could be true or false, depending on what degree of accuracy we are talking about. Since the paper is 8.5 inches wide, it is true that the paper approximately 7 inches wide, ±2 inches, but it is false to say that the paper is approximately 7 inches wide, ±1 inch.

The "error" in an approximation is the difference between the actual value and the approximate value:

$$\text{error} \ = \ \text{actual value} \ - \ \text{approximate value}$$

If the error is positive, that means that the actual value is greater than the approximate value, i.e. the approximate value is an underestimate. If the error is negative, that means that the approximate value is an overestimate. The "absolute error" is the absolute value of the error. (So it is always positive.)

If we do not know the actual value, it is impossible to calculate the exact error. However, we are often interested in bounds on the absolute error. For example, if the absolute error in a certain measurement is less than or equal to .5 cm, then the measurement is accurate up to ±.5 cm.

Numerical analysis (a topic beyond the scope of this course) gives formulas for bounds for the errors of the trapezoidal and midpoint approximations.

Exercises

4. What are the error bounds for the trapezoidal and midpoint approximations? (They are in a red box in the text. Make sure you copy down the words and not just the formulas.)

5. We will discuss the error in the trapezoidal and midpoint approximations of $\int_0^1 \cos(x^2)\,dx$. (See Examples 2 and 3.)

   (a) Let $f(x) = \cos(x^2)$. Find $f'(x)$ and $f''(x)$.

   (b) Use technology to graph $f''(x)$ on the domain $[0, 1]$. Sketch the graph below.

   (c) Use your sketch in (b) to find a bound for $|f''(x)|$. (Find a number K such that $|f''(x)| < K$.)

   (d) Find bounds for the errors in T_4 and M_4.

6. Again we consider $\int_0^1 \cos(x^2)\,dx$.

   (a) Find bounds for the errors in the approximations T_n and M_n, using the formulas in Exercise 4. (These bounds will be in terms of n. See Example 2.)

   (b) How large should we take n in order to guarantee that T_n is within 0.0001?

   (c) How large should we take n in order to guarantee that M_n is within 0.0001?

7. We continue to consider $\int_0^1 \cos(x^2)\,dx$.

   (a) Use Simpson's rule with $n = 4$ to estimate the integral. (See Example 4.)

   (b) Find the fourth derivative of $\cos(x^2)$.

   (c) Use a graph to find a bound, K, for the absolute value of the fourth derivative of $\cos(x^2)$ on the interval $[0, 1]$. Use the formula for the error bound for Simpson's Rule to bound the absolute error in your approximation from part (a).