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Main Points:

1. Reason by analogy with fractions
2. Linear factors vs. irreducible quadratic factors
3. Rationalization

Overview:

The method of **partial fractions** is an algebraic technique that can be helpful for integration. In particular, the partial fractions decomposition is a way to *rewrite* a rational function as a *sum* of *simpler* rational functions, as long as the degree of the numerator is smaller than the degree of the denominator. (If the degree of the denominator is larger than the degree of the numerator, long division of polynomials can be used first.) It is a reverse process to adding rational functions, and as such requires "undoing the common denominator."

We use the partial fractions decomposition to rewrite rational integrands as sums of simpler rational functions. To evaluate these simpler integrals it may be necessary to use a substitution. Recall some basic antiderivatives:

$$\int \frac{dx}{x} = \ln|x| + C\,; \qquad \int \frac{dx}{x^p} = \frac{1}{(1-p)\,x^{p-1}} + C\,, \quad (p > 1)\,; \qquad \int \frac{dx}{1+x^2} = \arctan(x) + C$$

1. Using a Partial Fractions Decomposition (PFD)

Exercise

1. In this exercise we will see how to *use* a partial fractions decomposition to rewrite an integral as a sum of two simpler ones. (Later we will see how to *find* a partial fractions decomposition.)

   (a) Verify that $\dfrac{2}{x+5} - \dfrac{1}{x-2} = \dfrac{x-9}{x^2+3x-10}$. (This is a partial fractions decomposition.)

(b) Evaluate the integrals:

i. $\displaystyle\int \frac{2}{x+5}\,dx$

ii. $\displaystyle\int \frac{1}{x-2}\,dx$

iii. $\displaystyle\int \frac{x-9}{x^2+3x-10}\,dx$ (Hint: combine your work from the previous two parts!)

2. PFD: Analogy With Fractions

How do we find the PFD for a rational function? What should it look like? Well, it should be a sum of "simpler" rational functions. But what do we mean by "simpler"? Before answering these questions we explore the analogous situation with fractions.

Exercises

2. Rewrite each fraction below as a sum of "simpler" fractions. In this case, "simpler" means that the denominator should be prime number or a power of a prime number.

   (a) Since $6 = 2 \cdot 3$, it makes sense to try and write $1/6$ in terms of halves and thirds, i.e. to try and find numerators so that $1/6 = ?/2 + ?/3$. Use guess-and-check to carry this out.

   (b) Since $18 = 2 \cdot 9 = 2 \cdot 3^2$, we could try to write $17/18 = ?/2 + ?/3$ too. Try this.

(c) Maybe it would work to write $17/18 = ?/2 + ?/3 + ?/9$ instead. Try this.

(d) What about an improper fraction like 7/6? Write it as a whole number plus some fractions: $7/6 = ? + ?/2 + ?/3$.

3. We will find the partial fractions decomposition for $\dfrac{2x+3}{x^2-4x-5}$.

   Notice that the denomniator factors as $(x+1)(x-5)$. This is analogous to (2a); the partial fractions decomposition is of the form

$$\frac{2x+3}{x^2-4x-5} = \frac{A}{x+1} + \frac{B}{x-5}$$

   (a) Clear denominators by multiplying both sides by $(x+1)(x-5)$.

   (b) Plug in $x = -1$ and solve for A.

   (c) Plug in $x = 5$ and solve for B.

   (d) There is another way of finding A and B which takes longer, but is more reliable in general. It is called "the method of undetermined coefficients."

   Go back to your equation in (a), expand/foil the right hand side, and collect terms.

You should get: $(A+B)x + (B-5A)$. This means that

$$2x+3 = (A+B)x + (B-5A)$$

The only way for this to be true is if $2 = A+B$ and $3 = B-5A$. Use these two equations to solve for A and B. Double-check your answers by comparing to (b) and (c).

4. We will find the partial fractions decomposition for $\dfrac{x-9}{(x+5)(x-2)^2}$.

   This is analogous to (2b) and (2c). Since there is a repeated linear factor, the partial fractions decomposition is of the form:

$$\frac{x-9}{(x+5)(x-2)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

   (a) Clear denominators by multiplying both sides by $(x+5)(x-2)^2$.

   (b) Solve for A and C by plugging in specific numbers, as in (3b) and (3c). Why can't you solve for B this way?

   (c) Use the method of undetermined coefficients to find A, B, and C.

Sometimes it is necessary to use long division of polynomials before finding the partial fractions decomposition. (This is analogous to an improper fraction as in (2d).) See Example 1.

3. Partial Fractions Decompositions Involving Irreducible Quadratic Factors

So far we have dealt with rational functions whose denominators factor into linear factors, but we know that this is not always going to be the case, because some polynomials, e.g. x^2+1, do not factor into linear factors. (Using the quadratic formula gives complex roots.)

It is true, however, that all polynomials factor into linear factors and irreducible quadratic factors. (This fact is not trivial, but we will take it for granted.)

Exercise

5. We will find the partial fractions decomposition for $\dfrac{x-9}{(x+5)(x^2+1)}$.

   (a) In this case the "prime factors" in the denominator are $(x+5)$ and (x^2+1). What happens when you try to find numbers A and B such that

   $$\frac{x-9}{(x+5)(x^2+1)} = \frac{A}{x+5} + \frac{B}{x^2+1} \ ?$$

   (b) The issue is the (x^2+1) is a different kind of "prime" than $x+5$. Instead of having just a number like B in the numerator, we need something of the form $Bx+C$. Use the method of undetermined coefficients to find A, B, and C.

   $$\frac{x-9}{(x+5)(x^2+1)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+1}$$

We deal with powers of irreducible quadratics analogously to powers of linear factors. For example,

$$\frac{x-9}{(x+5)(x-2)^2(x^2+1)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

is the correct form of the partial fractions decomposition of the rational function on the left side.

Exercise

6. Write out the form of the partial fraction decomposition of the function. (See Example 7). Do not determine the numerical values of the coefficients.

   (a) $\dfrac{10}{5x^2-2x^3}$

   (b) $\dfrac{x^2}{x^2+x+2}$

   (c) $\dfrac{1}{(x^2-9)^2}$

4. Rationalization

Sometimes a substitution transforms an integrand that is *not* rational into an integrand that *is* rational. In this case, following the substitution with a PFD can be a useful technique. See Example 9.