Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

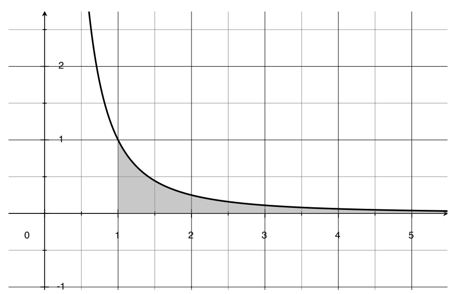
Main Points:

1. Two types of improper integrals
2. An important family of examples: $f(x) = \frac{1}{x^p}$
3. Comparison Theorem: an indirect method for determining whether an improper integral converges
4. A careful argument: set-up, check hypotheses, apply theorem, draw conclusion

1. Integrating over infinite intervals

Exercises.

1. Suppose we wish to determine the area of the infinite region bounded by the curves $y = \frac{1}{x^2}$, $x = 1$, and the x-axis:



(Of course, we ought to be wondering if it is reasonable for an infinite region to have finite area . . .)

(a) Evaluate the following integrals:

i. $\int_1^{10} \frac{1}{x^2}\,dx$

ii. $\int_1^{100} \frac{1}{x^2}\,dx$

iii. $\int_1^{1000} \frac{1}{x^2}\,dx$

(b) Based on your answers in (a), do you think the area of the infinite region described above has a finite area? If so, make a guess for what the area is.

(c) Now consider an arbitrary number T greater than one, and let $A_T = \int_1^T \frac{1}{x^2}\,dx$. (We considered $T = 10, 100, 1000$ above.)

i. What is the value of A_T, for arbitrary T?

$A_T \;=\; \int_1^T \frac{1}{x^2}\,dx \;=$

ii. What is the limit of A_T as T tends to infinity?

$\lim\limits_{T\to\infty} A_T \;=$

(d) The area of the infinite region is $A = \lim\limits_{T\to\infty} A_T$ from (c)(ii). How does this compare to your guess in (b)?

In general, integrating over infinite intervals is defined by taking a limit after integrating over larger and larger finite intervals, as you did in Exercise 1:

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{T\to-\infty} \int_T^b f(x)\,dx$$

Exercises

2. (a) Look at Definition 1 in the textbook. (It's in a big red box.) What does it mean for an improper integral to be convergent? Divergent?

(b) Give the definition (in the same red box) of the doubly-infinite integral:

$\int_{-\infty}^{\infty} f(x)\,dx \;=$

3. (a) Read Example 1. Is $\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent? ______________________

   (b) Use a limit to determine whether $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is convergent or divergent. If it converges, what is its value?

   (c) Read Example 4, summarize the result, and explain how (a) and (b) fit the pattern.

4. Determine if the integral is convergent or divergent. If convergent, what is its value?

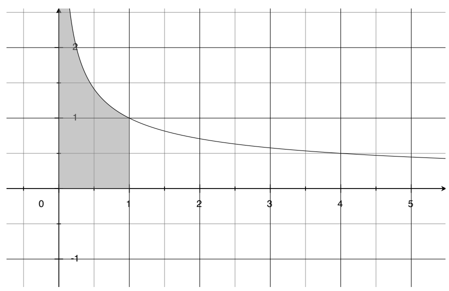
   (a) $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

   (b) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

2. Integrating up to or over discontinuities

Exercises

5. Suppose we wish to determine the area of the infinite region bounded by the curves $y = \frac{1}{\sqrt{x}}$, $x = 1$, and the y-axis:



(a) Evaluate the following integrals:

i. $\int_{0.1}^{1} \frac{1}{\sqrt{x}}\, dx$

ii. $\int_{0.01}^{1} \frac{1}{\sqrt{x}}\, dx$

iii. $\int_{0.001}^{1} \frac{1}{\sqrt{x}}\, dx$

(b) Based on your answers in (a), do you think the area of the infinite region described above has a finite area? If so, make a guess for what the area is.

(c) Now consider an arbitrary number t between zero and one, and let $A_t = \int_{t}^{1} \frac{1}{\sqrt{x}}\, dx$. (We considered $t = 0.1, 0.01, 0.001$ above.)

i. What is the value of A_t, for arbitrary t?

$$A_t = \int_{t}^{1} \frac{1}{\sqrt{x}}\, dx =$$

ii. What is the limit of A_t as t approaches zero from above?

$$\lim_{t \to 0^+} A_t =$$

(d) The area of the infinite region is $A = \lim_{t \to 0+} A_t$ from (c)(ii). How does this compare to your guess in (b)?

In general, an integral over an interval containing a discontinuity is defined by taking a limit, as you did in Exercise 5:

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \quad \text{(discontinuity at } a\text{)}$$
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx \quad \text{(discontinuity at } b\text{)}$$

Exercises

6. (a) Suppose $p > 1$. Evaluate $\int_0^1 \frac{1}{x^p}\,dx$, or show that the integral diverges.

(b) Now suppose $0 < p < 1$. Evaluate $\int_0^1 \frac{1}{x^p}\,dx$ or show that the integral diverges.

(c) Finally, evaluate the same integral, with $p = 1$, i.e. $\int_0^1 \frac{1}{x}\,dx$, or show that it diverges.

7. If a discontinuity occurs in the middle of the interval over which we are integrating, special care is needed. Suppose a function $f(x)$ has a discontinuity at a point $x = c$ in the middle of the interval $[a, b]$. See Definition 3, and give the definition of the following improper integral:

$$\int_a^b f(x)\,dx =$$

8. Evaluate the integral $\int_{-1}^{8} x^{-1/3}\,dx$. (You need to break up the integral into two integrals, and use limits to evaluate each integral, as in Example 7.)

3. Comparison Theorem: Introduction

Suppose we wish to determine whether or not $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges. Recall that, if it converges, it is:

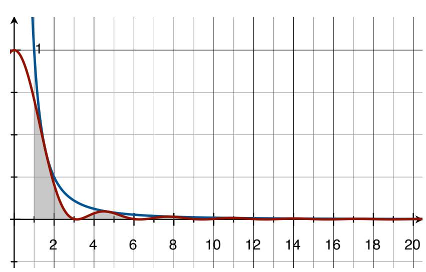
$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx \;=\; \lim_{T\to\infty} \int_1^T \frac{\sin^2(x)}{x^2}\,dx$$

If we had an antiderivative for $\frac{\sin^2(x)}{x^2}$, we could use the FTC, and then evaluate the limit, as in the problems discussed last time. However, it is not at all obvious how to find an antiderivative in this case, so we take an alternate route: we compare to a function whose antiderivative we do know: $\frac{1}{x^2}$.

We use the fact that $\sin^2 x$ is bounded between 0 and 1 to compare $f(x)$ and $g(x)$:

$$0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \;\le\; \frac{\sin^2(x)}{x^2} \;\le\; \frac{1}{x^2}$$

Thus is is reasonable to conclude that the area under the curve $y = \frac{\sin^2(x)}{x^2}$ is less than the area under the curve $y = \frac{1}{x^2}$, as is illustrated in the graph below.



The fact that that $\int_1^\infty \frac{1}{x^2}\,dx$ is finite implies that $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is finite, i.e. the integral converges. This is the idea behind the Comparison Theorem.

Exercises.

9. Do you think that $\int_1^\infty \frac{\cos(x)+1}{2x^3}\,dx$ converges or diverges? Explain your reasoning. (Hint: Use the fact that $-1 \leq \cos(x) \leq 1$. Look at a graph if you need to.)

4. Using the Comparison Theorem

To justify the guesses we have made above, we need to use the Comparison Theorem. Whenever you want to use a theorem to draw a conclusion, you need to: (1) set up the theorem, (2) check that the hypotheses of the theorem are satisfied, (3) cite the theorem and draw your conclusion.

Exercises

10. Read the Comparison Theorem in the textbook and state it here.

11. In this problem, you will give a careful argument justifying our guess that $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges.

    (a) (Set-up.) There are three things to specify: f, g, and a. State what each is in this example.

    (b) (Check hypotheses.) There are several things to check: (1) that f and g are continuous functions, (2) that $f(x) \geq g(x) \geq 0$ for all $x \geq a$, and (3) that $\int_a^\infty f(x)dx$ is convergent. Explain why all of these are true in this example.

(c) (Cite theorem and draw conclusion.) Here you merely need to say, "By the Comparison Theorem, we can conclude that . . . " (Finish this sentence.)

12. Give a careful argument to justify your guess in Exercise 9. (Use the outline given in Exercise 11.)