Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Arc length of graph of $y = f(x)$
2. Arc length of graph of $x = g(y)$.

To find the length of a curve, we may divide the curve into n small pieces, approximate each piece with a line segment (part of the secant line), find the length of each line segment, add up the lengths, and then take a limit as $n \to \infty$. This process gives an integral for the arc length of the curve.

The key is that the length Δs of a small piece of the curve is approximated by:

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x = \sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2 + 1} \cdot \Delta y .$$

Using differentials, we may express this as:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy .$$

Exercise.

1. We will find the arc length of the part of the curve $y = \frac{2}{3}x^{3/2}$ corresponding to $0 \le x \le 2$.

   (a) Find $\frac{dy}{dx}$, and use this to find a formula for the differential ds.

   (b) Integrate ds to find the arc length of the part of the curve corresponding to $0 \le x \le 2$.

When the curve is given by $y = f(x)$ for $a \leq x \leq b$ and $f'(x)$ is continuous on $[a, b]$, the arc length is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx .$$

Similarly if the curve is given by $x = g(y)$ for $c \leq y \leq d$, the arc length is:

$$L = \int_c^d \sqrt{(g'(y))^2 + 1}\, dy ,$$

as long as $g'(y)$ is continuous on $[c, d]$.

Exercises.

2. We will find the arc length of the part of the curve $x = \frac{y^4}{8} + \frac{1}{4y^2}$ corresponding to $1 \leq y \leq 2$. (See Example 2 in the textbook.)

   (a) Find $\frac{dx}{dy}$, and use this to find a formula for the differential ds.

   (b) Find the arc length of the part of the curve corresponding to $1 \leq y \leq 2$.

3. We will find the arc length of the part of the curve $y^3 = x^2$ from $(0, 0)$ to $(1, 1)$.

   (a) Use technology to help you sketch a graph of this curve for $-2 \leq x \leq 2$.

   (b) Set up **two** integrals for the arc length of $y^3 = x^2$ from $(0, 0)$ to $(1, 1)$.

   (c) One of the integrals in 3b should be an improper integral. Which one?

(d) Evaluate both integrals in 3b.

Hint. For one of the integrals you may want to rewrite the integrand like this:

$$1 + \tfrac{a^2}{b^2} x^{-2/3} \quad = \quad 1 + \frac{a^2}{b^2 x^{2/3}} \quad = \quad \frac{b^2 x^{2/3} + a^2}{b^2 x^{2/3}} \quad = \quad \frac{b^2 x^{2/3} + a^2}{(b x^{1/3})^2}\ .$$