Math 132, Prep 11.1 (Sequences)

Name: ______________________________

Names of collaborators: ______________________________

Main Points:

1. sequences: various notations
2. convergence/divergence of sequences
3. monotone and bounded sequences

1. Sequences

A sequence is a list of numbers in a definite order. There are several ways of denoting a sequence, but the simplest involves curly braces and elipses. Some examples of sequences denoted in this way are:

$$\{1, 2, 3, 4, 5, 6, \dots\} \qquad \{1, -1, 1, -1, 1, -1, \dots\} \qquad \{1,\ 1/2,\ 1/4,\ 1/8, \dots\}$$

Another way of denoting a sequence is by giving a formula for the n^{th} term of the sequence. For example the three sequences above could be represented with the following three formulas:

$$a_n = n\ ,\ \ n \geq 1 \qquad b_n = (-1)^n\ ,\ \ n \geq 0 \qquad c_n = 1/2^n\ ,\ \ n \geq 0$$

See Examples 1 and 2.

Exercises.

1. The formula for the n^{th} term of a sequence is given. Use the formula to find the first five terms of the sequence, and write the sequence in the notation with curly braces and elipses.

   (a) $a_n = \dfrac{n}{n+1}\ ,\ \ n \geq 2$

   (b) $b_n = \dfrac{2n}{n^2+1}\ ,\ \ n \geq 0$

   (c) $a_n = \dfrac{(-1)^n\, n}{n!+1}\ ,\ \ n \geq 1$ (Hint: remember that $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.)

2. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

   (a) $\{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots\}$

   (b) $\{4, -7, 10, -13, 16, \dots\}$

   (c) $\{6, 9/4,\ 4/3,\ 15/16, 18/25, 21/36 \dots\}$

Some sequences are more easily described **recursively**. See Example 3(c) and the beginning of Example 14.

Exercises

3. List the first five terms of the following recursively defined sequences.

   (a) $a_1 = 6$, and $a_{n+1} = a_n/n$, for $n \geq 1$.

   (b) $b_0 = 1$, $b_1 = 2$, and $b_{n+1} = 2b_{n-1} + b_n$, for $n \geq 1$.

2. The Limit of a Sequence

Informally, if the numbers a_n approach a specific, finite number L as $n \to \infty$, then the sequence is said to **converge**, and L is called the **limit** of the sequence. If a sequence does not have a limit, it is said to **diverge**. See the textbook for a more careful discussion of the limit of a sequence.

Exercises.

4. There are several useful theorems we can use for finding limits of sequences.

   (a) Copy down Theorem 4 and explain how it is used in Example 6.

   (b) Copy down Theorem 6 and explain how it is used in Example 8.

   (c) Copy down Theorem 7 and explain how it is used in Example 9.

   (d) Copy down the Squeeze Theorem for Sequences (in a red box, before Theorem 6) and explain how it is used in Example 10.

(e) Read Example 11, and summarize the result (stated in the box at the end of the example.)

5. Determine whether the following sequences converge or diverge. If convergent, find the limit.

   (a) $a_n = \dfrac{3+5n^2}{n+n^2}$ (Hint: divide top and bottom by the highest power of n in the denominator.)

   (b) $a_n = e^{2n/(n+2)}$ (Hint: what happens to $2n/(n+1)$ as $n \to \infty$?)

   (c) $a_n = \dfrac{3^{n+1}}{5^n}$ (Hint: find the common factor.)

(d) $a_n \ = \ 2^{-n} \cos(n\pi)$ (Hint: Take the absolute value.)

(e) $a_n \ = \ n \sin(1/n)$ (Hint: Find a matching function $f(x)$ and use L'Hospital's rule.)

3. The Monotone Boundedness Theorem

A sequence a_n is **increasing** if the values of successive terms are larger and larger, i.e. $a_{n+1} > a_n$ for all n for which the sequence is defined. An example of an increasing sequence is $a_n = 2^n$. A sequence is **decreasing** if the values of successive terms are smaller and smaller. An example of a decreasing sequence is $a_n = (1/2)^n$. A sequence is **monotonic** if it is **either** increasing or decreasing.

6. Determine whether the sequence is increasing, decreasing, or neither (so not monotonic.)

   (a) $a_n = \cos(n), \quad n \geq 0$

   (b) $b_n = \frac{1}{n}, \quad n \geq 1$

A sequence a_n is **bounded above** if there is some number M, which no a_n exceeds, i.e. $a_n \leq M$ for all n. Similarly a sequence a_n is **bounded below** if there is a number m, which no a_n is less than, i.e. $a_n \leq m$ for all n. When a sequence is bounded above and bounded below, we say that it is **bounded**.

7. For each sequence in the previous problem, determine whether or not it is bounded.

The **Monotone Boundedness Theorem** can be used to show that certain sequences are convergent (without having to find the limit).

8. State Theorem 12.