Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. comparing with familiar series, whose convergence/divergence is known
2. making a careful argument and invoking the Direct or Limit Comparison Test

When our intuition tells us that a certain series ought to converge (or diverge) because it is similar to a familiar series, one that we *know* converges (resp. diverges), we can sometimes use a **comparison test** to give a careful argument in support of our intuition. In this section, we discuss two comparison tests: the Direct Comparison Test and the Limit Comparison Test.

For example, we may *suspect* that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$$

converges, since it is similar to a p-series with $p = 2$. We will use the Direct Comparison Test to justify this.

Note: Recall that we can use the Integral Test to determine whether or not a series converges; it is not necessary to have an intuition ahead of time about whether or not the series converges. In contrast, to use a comparison test, you must first have a hunch about whether the series converges or diverges. A comparison test is then used to give a careful argument (hopefully) proving your hunch to be correct.

1. Using the Direct Comparison Test

Exercises.

1. Read the beginning of Section 11.4, up to but not including Example 1.

   (a) State the Direct Comparison Test.

   (b) To be able to use a comparison test, it is helpful to have some familiar series to compare with. Give two examples of simple families of series whose convergence/divergence is known to us. (See the paragraph after the proof of the Direct Comparison Test.)

2. In this exercise, we will use the Direct Comparison Test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges, by comparison with the p-series, with $p = 2$.

   (a) (**Set-up**) Define the series $\sum a_n$ to be the given series and $\sum b_n$ to be the other (more familiar) series, whose convergence/divergence is known.

   (b) (**Check Hypotheses**) Check to make sure the series $\sum a_n$ and $\sum b_n$ satisfy the hypothesis in the Direct Comparison Test, namely that they are series with positive terms.

   (c) (**Compare Terms**) Prove an inequality for the terms of the two series. (For convergence, show $a_n \leq b_n$ for all n, or, for divergence, show $b_n \leq a_n$ for all n).

   (d) (**Discuss known series**) Explain how you know that the more familiar series $\sum b_n$ converges or diverges.

   (e) (**Apply Test and Draw Conclusion**) Use the Direct Comparison Test to conclude that the series $\sum a_n$ converges (or diverges), and state your conclusion in a sentence. Make sure to include the phrase "by the Direct Comparison Test" somewhere in your sentence.

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3+4}$.

   (a) Do you think this series converges or diverges? (It converges.) What known (convergent) series can you compare it to? Can you prove an inequality like $a_n \leq b_n$ for the terms (a_n) of this series and the terms (b_n) of the familiar (convergent) series?

   (b) Give a careful argument, using the Direct Comparison Test, to prove that the series converges. (Your argument should follow the outline given in the previous problem.)

4. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$.

   (a) Do you think this series converges or diverges? What known series can you compare it to? Can you prove an inequality of terms?

(b) Give a careful argument, using the Direct Comparison Test, to prove that the series converges (or diverges.)

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - 1/2}$.

(a) Do you think this series converges or diverges? What known series can you compare it to? Can you prove an inequality of terms?

(b) Give a careful argument, using the Direct Comparison Test, to prove that the series converges (or diverges.)

2. Using the Limit Comparison Test

The Limit Comparison Test can be used in some cases where the Direct Comparison Test cannot be used. For example, we may *suspect* that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1}$$

converges, since it is similar to a p-series with $p = 2$. However, the Comparison Test cannot be used to prove this, since the terms of this series are not actually *smaller* than the terms of this p-series. We will use the Limit Comparison Test instead.

Exercises.

1. Read about the Limit Comparison Test (after Example 2 ends). State the Limit Comparison Test (in a red box.)

2. In this exercise, we will use the Limit Comparison Test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1}$ converges, by comparison with the p-series, with $p = 2$.

   (a) (**Set-up**) Define the series $\sum a_n$ to be the given series and $\sum b_n$ to be the other (more familiar) series, whose convergence/divergence is known.

   (b) (**Check Hypotheses**) Check to make sure the series $\sum a_n$ and $\sum b_n$ satisfy the hypothesis in the Limit Comparison Test, namely that they are series with positive terms.

   (c) (**Set up the ratio and take the limit.**) Set up the ratio a_n/b_n and simplify. Then take the limit as $n \to \infty$.

   $$\frac{a_n}{b_n} =$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad =$$

(d) Is this limit a finite, positive number? ________

If so, proceed to the next step, if not you need to find a different sequence b_n.

(e) (**Discuss known series**) Explain how you know that the more familiar series $\sum b_n$ converges or diverges.

(f) (**Apply Test and Draw Conclusion**) Use the Limit Comparison Test to conclude that the series $\sum a_n$ converges (or diverges), and state your conclusion in a sentence. Make sure to include the phrase "by the Limit Comparison Test" somewhere in your sentence.

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$.

(a) Do you think this series converges or diverges? (It diverges.) What known (divergent) series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number?

(b) Give a careful argument, using the Limit Comparison Test, to prove that the series diverges. (Your argument should follow the outline given in the previous problem.)

4. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n-1}}$.

   (a) Do you think this series converges or diverges? What known series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number? (Hint: Remember that $n\sqrt{n} = n^{3/2}$.)

   (b) Give a careful argument, using the Limit Comparison Test, to prove that the series converges (or diverges.)

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$.

   (a) Do you think this series converges or diverges? What known series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number? (Hint: Notice that $e^{1/n} \to e^0 = 1$ as $n \to \infty$.)

(b) Give a careful argument, using the Limit Comparison Test, to prove that the series converges (or diverges.)

6. Look back at the four series above. Could you have used the Comparison Test instead of the Limit Comparison Test for any of them? Which one(s)?