Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. absolute, conditional convergence
2. ratio test, root test

1. Absolute Convergence

Given a series $\sum a_n$ with positive and negative terms, we may consider the related series $\sum |a_n|$. It is a non-trivial, but true, fact that if this series converges, then the original series must converge. In this case we say that the original series converges *absolutely*. If a series converges, but not absolutely, we say that it converges *conditionally*.

To recap: a series $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges, but $\sum a_n$ converges only *conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

An example of an alternating series that is absolutely convergent series is a geometric series with common ratio r in the range $-1 < r < 0$, like:

$$\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^{n-1} = 1 - \frac{2}{3} + \frac{4}{9} - \frac{16}{27} + \dots$$

An example of a conditionally convergent series is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Exercises.

1. Consider the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3}{5}\right)^{n-1}$.

   (a) Does this series converge? Why or why not?

   (b) Now consider the related series $\sum |a_n|$ (where a_n are the terms of the original series.) Does this series converge? Explain.

   (c) Is the original series absolutely convergent, conditionally convergent, or divergent?

2. Consider the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-6}{5}\right)^{n-1}$.

   (a) Does this series converge? Why or why not?

   (b) Now consider the related series $\sum |a_n|$ (where a_n are the terms of the original series.) Does this series converge? Explain.

   (c) Is the original series absolutely convergent, conditionally convergent, or divergent?

3. Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$.

   (a) Does this series converge? (Use the Alternating Series Test.)

   (b) Now consider the related series $\sum |a_n|$ (where a_n are the terms of the original series.) Does this series converge?

   (c) Is the original series absolutely convergent, conditionally convergent, or divergent?

2. The Ratio Test

Recall that a geometric series converges if the ratio of successive terms (which is a constant, called the common ratio) has absolute value less than one. For a series $\sum a_n$ that is not geometric, the ratio a_{n+1}/a_n of successive terms will not be a constant, but if the absolute value of the ratio *approaches* a constant *less than one* as n increases, the series converges. This is the idea behind the Ratio Test.

Exercises

4. State the Ratio Test (in a red box at the beginning of Section 11.6.)

5. Determine whether the following series converge absolutely, converge conditionally, or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} e^{-n}\, n!$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$

(Hint: Instead of using the Ratio Test, show $\sum a_n$ converges absolutely by considering $\sum |a_n|$ and using the Direct Comparison Test to show that this series converges.)

3. The Root Test

The Root Test is another test for absolute convergence that takes inspiration from the geometric series: if $\sum ar^n$ is a geometric series, with both a and r positive, the the nth root of the nth term is $\sqrt[n]{a} \cdot r = a^{1/n} \cdot r$. As $n \to \infty$, the exponent $\frac{1}{n} \to 0$, so $\sqrt[n]{a} \to 1$, since a is a nonzero constant. Thus the limit of the nth root of the nth term is precisely the common ratio r (which we assumed to be positive). We look at whether r is greater or less than 1 to determine whether the geometric series diverges or converges. In the Root Test, we look at the nth root of the absolute value of the nth term to help us determine whether the series converges (absolutely) or diverges.

6. State the Root Test (in a red box, after the end of Example 3.)

7. Determine whether the following series converge absolutely, converge conditionally, or diverge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln(n))^n}$