**Math 132, Prep 11.9 (Representing Functions as Power Series)**

**Name:** ______________________ **Section:** ________

**Names of collaborators:** ______________________

## Main Points:

1. use geometric series to represent a given rational function with a power series
2. using sigma notation, changing indices
3. differentiate and integrate power series
4. express derivatives/antiderivatives of rational functions as power series

Recall that $\sum_{n=0}^{\infty} x^n$ is a function with domain $(-1, 1)$. We can find a rational function that agrees with the power series on its domain by remembering that the sum of a geometric series with initial term $a$ and common ratio $r$ is $a/(1-r)$. Thus

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

We can use this fact to find power series representations of some functions that are similar to this one. For example, with $a = 3$ and $r = 4x$,

$$\frac{3}{1-4x} = \sum_{n=0}^{\infty} 3 \cdot (4x)^n = \sum_{n=0}^{\infty} 3 \cdot 4^n\, x^n = 3 + 12x + 48x^3 + \ldots$$

This series will converge when $|r| < 1$, i.e. when $|4x| < 1$, i.e. when $|x| < 1/4$. Thus the interval of convergence of the power series is $(-1/4, 1/4)$.

See Examples 1, 2, and 3 in the textbook for more examples of how to find a power series representation of a function, using the geometric series.

**Exercises.**

1. Consider the function $f(x) = \dfrac{2}{1-3x}$.

   (a) Find $a$ and $r$ such that $f(x) = a/(1-r)$.

   $a =$ $r =$

   (b) Write $f(x)$ as a power series using the sigma notation and in expanded form (with ellipsis.)

   (c) Find the radius and interval of convergence of the power series.

2. Find a power series representation and the radius of convergence for each of the following:

   (a) $g(x) = \dfrac{x}{1+x^2}$

   (b) $h(x) = \dfrac{1}{2-x}$

   (c) $F(x) = \dfrac{5}{1-4x^2}$

## 2. Differentiating and Integrating Power Series

Recall that we can consider a power series as polynomial with an infinite number of terms. For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

and

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} + \ldots + \frac{(-1)^n\, x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \ldots$$

One might hope that, to differentiate (or integrate) a power series, we can simply differentiate (or integrate) like polynomials, and, in fact, this is true! This is one of the main reasons power series are such a useful way to represent a function.

One also might be concerned (and rightly so!) about the convergence of a power series obtained by differentiation (or integration). It turns out that the radius of convergence does not change, but that the behavior at the endpoints of the interval of convergence may change.

**Exercises.**

3. In this exercise we differentiate and integrate $f(x) = \sum_{n=0}^{\infty} x^n$.

   (a) Write out $f(x)$ as a polynomial with infinitely many terms, then differentiate term by term to find $f'(x)$.

   (b) Write the derivative $f'(x)$ in sigma notation.

   (c) Again, write out $f(x)$ as a polynomial with infinitely many terms, and, this time, integrate term by term to find the antiderivatives of $f(x)$.

(d) Write the antiderivatives of $f$ in sigma notation.

4. In this exercise we differentiate and integrate $g(x) = \displaystyle\sum_{n=0}^{\infty} 3^n x^{2n}$.

(a) Write out $g(x)$ as a polynomial with infinitely many terms, then differentiate term by term to find $g'(x)$.

(b) Write the derivative $g'(x)$ in sigma notation.

(c) Again, write out $g(x)$ as a polynomial with infinitely many terms, and, this time, integrate term by term to find the antiderivatives of $g(x)$.

(d) Write the antiderivatives of $g$ in sigma notation.

## 3. Finding More Power Series Representations

We can now use differentiation and integration to allow us to find power series representations for more functions. See Examples 5-7 in the textbook.

**Exercises.**

5. Recall that the power series in Problem 3 can also be written as $f(x) = \dfrac{1}{1-x}$ for $|x| < 1$.

   (a) Find the derivative of $f(x) = \dfrac{1}{1-x}$.

   (b) Find $\int f(x)dx$.

   (c) Find a power series representation for $\dfrac{1}{(1-x)^2}$. (Hint: Use Problem 3.) What is the radius of convergence? What is the interval of convergence?

   (d) Find a power series representation for $-\ln(1-x)$. What is the radius of convergence? What is the integral of convergence?

6. In this exercise we return to the power series in Problem 4, $g(x) = \sum_{n=0}^{\infty} 3^n x^{2n}$.

   (a) Use the formula for the sum of a geometric series to write $g(x)$ as a rational function.

   (b) Find the derivative of the rational function $g(x)$.

   (c) Find a power series representation for $\dfrac{6x}{(1-3x^2)^2}$. What is the radius of convergence? The interval of convergence?