Math 132, Prep 11.9B, 11.10 (Taylor Series and Applications to Integration)

Name: ____ **Section:** ________

Names of collaborators: ____

Main Points:

1. Recall the notion of a Taylor series
2. New Taylor series from old
3. Using power series to evaluate indefinite integrals or estimate definite integrals

1. Taylor Series

Recall our study of **Taylor polynomials** at the beginning of the semester. The first three Taylor polynomials for a function $f(x)$ centered at a given point $x = a$ are simply the constant approximation: $T_0(x) = f(a)$, the linear approximation $T_1(x) = f(a) + f'(a)(x-a)$, and the quadratic approximation $T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$. In general the Taylor coefficient c_n for the $(x-a)^n$ term in the Taylor polynomial is $c_n = f^{(n)}(a)/n!$, where $f^{(n)}$ refers to the nth derivative of f, so

$$T_n(x) \;=\; f(a) \;+\; f'(a)(x-a) \;+\; \frac{f''(a)}{2!}(x-a)^2 \;+\; \frac{f'''(a)}{3!}(x-a)^3 \;+\; \dots \;+\; \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Each successive Taylor polynomial gives a better approximation for $f(x)$ near $x = a$. We can describe the family of *all* Taylor polynomials for a given function centered at a given point using **Taylor series**.

Note that a Taylor series centered at $x = 0$ is called a **Maclaurin series**.

Now that we have studied infinite series and power series in particular, we are in a position to discuss Taylor series in more depth and with more rigor. Theorem 8 tells us the condition that ensures the Taylor series of a function is a valid representation of the function (essentially, when the remainder goes to zero.) Now that we have a precise understanding of convergence, we can find the radius of convergence of a Taylor series analytically (using the Ratio Test).

We take this as an opportunity to review the Taylor series for e^x, $\sin x$, and $\cos x$ and to discuss how to find new Taylor series from ones we already know.

Exercises.

1. See Table 1, in Section 11.10. What are the Maclaurin series for e^x, $\sin(x)$, and $\cos(x)$?

2. (a) Find the first four Taylor polynomials of $g(x) = \frac{1}{x}$ centered at $x = -3$.

   (b) Find the Taylor series of $g(x)$ centered at $x = -3$. What is the radius of convergence?

3. We can use known Taylor series to find Taylor series for related functions. (See Example 10.)

   (a) Use the Taylor series for $\ln(1 + x)$ to find a Taylor series for $x \ln(1 + x)$.

(b) Use the Taylor series for e^x to find a Taylor series for $\dfrac{e^x - 1}{x}$.

Hint. Since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, subtracting 1 gives: $e^x - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

(c) Use the Taylor series for $\cos(x)$ to find a Taylor series for $\cos(x^2)$.

Hint. Substitute $u = x^2$ in for x in the Taylor series for $\cos x$.

(d) Use the Taylor series for e^x to find a Taylor series for $e^{-x/2}$.

2. Applications to Integration

Recall that not every elementary function has an elementary antiderivative. For example, the antiderivative of $\cos(x^2)$ is a certain Fresnel function, and it cannot be expressed in terms of elementary functions. However, now that we know how to find a power series representation for $\cos(x^2)$ and we know how to integrate power series, we have a way to find a power series representation for the antiderivative of $\cos(x^2)$, and we can use it to estimate definite integrals of $\cos(x^2)$.

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\cos(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots \\
\int \cos(x^2)\, dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n}}{(2n)!}\, dx = \int \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots\right) dx \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+1}}{(2n)!(4n+1)} = C + x - \frac{x^5}{2!\cdot 5} + \frac{x^9}{4!\cdot 9} - \frac{x^{13}}{6!\cdot 13} + \dots
\end{aligned}$$

Thus, for example,

$$\int_0^{0.6} \cos(x^2)\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n\, (0.6)^{4n+1}}{(2n)!(4n+1)}$$

and we can use a partial sum to estimate the integral. Since this is an alternating series, we can use the Alternating Series Estimation Theorem to determine the accuracy of our estimate. In particular,

$$s_2 = \sum_{n=0}^{2} \frac{(-1)^n\, (0.6)^{4n+1}}{(2n)!(4n+1)} = (0.6) - \frac{(0.6)^5}{2!\cdot 5} + \frac{(0.6)^9}{4!\cdot 9} = 0.5922710656\dots$$

estimates the value of the integral, accurate to six decimal places, since $b_3 = 1.39537 \times 10^{-7}$ will not change the sixth decimal place. Therefore

$$\int_0^{0.6} \cos(x^2)\, dx \approx 0.592271$$

Exercises.

4. Evaluate the indefinite integrals as power series. (See 11.9, Example 7a, and 11.10, Example 13a.)

   (a) $\displaystyle\int \frac{t}{1+t^3}\, dt$

(b) $\displaystyle\int x^2 \ln(1+x)\, dx$

(c) $\displaystyle\int \frac{e^t - 1}{t}\, dt$

(d) $\displaystyle\int x\cos(x^3)\, dx$

5. Estimate the definite integrals, accurate to the stated number of decimal places. (See 11.9, Example 7b, and 11.10, Example 13b.)

   (a) $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx$ (6 decimal places)

   (b) $\displaystyle\int_0^{1} \sin(x^4)\,dx$ (4 decimal places)