Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. rates of change in differential equations
2. qualitative analysis of differential equations
3. families of solutions of differential equations
4. verifying a solution

1. Rates of Change Expressed in Differential Equations

When a quantity A is *directly proportional* to a quantity B, that means that there is a positive constant, say k, such that $A = kB$. The constant k is called the *constant of proportionality.*

Recall that the rate of change of a quantity A with respect to a quantity t is the derivative of A with respect to t, in Leibnitz notation, $\frac{dA}{dt}$. Often, we are interested in the rate at which a quantity changes over time.

A simple model for population growth operates under the assumption that a population will grow at a rate directly proportional to the size of the population. In other words $\frac{dP}{dt}$ is proportional to P. How would we write this in an equation?

$$\frac{dP}{dt} = k\,P \qquad \text{(for some } k > 0\text{)}$$

This is an example of a differential equation, an equation relating an unknown function with one or more of its derivatives. Here the unknown function is P and the equation relates P with its first derivative.

Exercises.

1. (a) Radioactive substances decay at a rate proportional to the quantity present. Write a differential equation for the quantity, Q, of a radioactive substance present at time t. Is the constant of proportionality positive or negative?

   (b) A pollutant spilled on the ground decays at a rate of 8% a day. In addition, clean-up crews remove the pollutant at a rate of 30 gallons a day. Write a differential equation for the amount of pollutant, P, in gallons, left after t days.

   (c) Toxins in pesticides can get into the food chain and accumulate in the body. A person consumes 10 micrograms a day of a toxin, ingested throughout the day. The toxin leaves the body at a continuous rate of 3% every day. Write a differential equation for the amount of toxin, A in micrograms, in the person's body as a function of the number of days, t.

2. Qualitative Analysis of Differential Equations

A differential equation relates an unknown function with one or more of its derivatives. Any function that satisfies the differential equation is called a *solution* to the differential equation. Consider the simple differential equation:

$$\frac{dy}{dx} = \cos x + 1$$

A solution to this differential equation is a function y whose derivative (with respect to x) is $\cos x + 1$. In this case, we already know how to find y; it is an antiderivative for $\cos x + 1$. So, for example, $y = \sin x + x + 5$ is a solution, as is $y = \sin x + x - 11$. These are called *particular solutions.* In fact, for any real number C, the function $y = \cos x + 1 + C$ is a solution. As we can see, the differential equation does not have only one solution; it has an entire *family* of solutions.

We can learn a lot about the general shape of the graphs of solutions just by looking at the differential equation. Recall that a quantity y will increase (with respect to x) if $\frac{dy}{dx} > 0$, it will decrease if $\frac{dy}{dx} < 0$, and it will (perhaps momentarily) be neither increasing nor decreasing if $\frac{dy}{dx} = 0$.

Exercises

2. Consider the differential equation $\frac{dy}{dx} = y(y-1)(y-2)$.

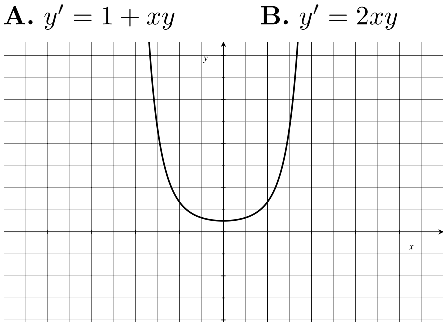
   (a) What are the constant solutions of the equation?

   (b) For what values of y is y increasing?

   (c) For what values of y is y decreasing?

3. The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.

   A. $y' = 1 + xy$ **B.** $y' = 2xy$ **C.** $y' = 1 - 2xy$



3. Families of solutions

Recall that a differential equation relates an unknown function and one or more of its derivatives. A solution to a differential equation is a function that satisfies the differential equation. Consider the simple differential equation:

$$\frac{dy}{dx} = \cos x + 1$$

A solution to this differential equation is a function y whose derivative (with respect to x) is $\cos x + 1$. In this case, we already know how to find y; it is an antiderivative for $\cos x + 1$. So, for example, $y = \sin x + x + 5$ is a solution, as is $y = \sin x + x - 11$. These are called *particular solutions.* In fact, for any real number C, the function $y = \cos x + 1 + C$ is a solution. As we can see, the differential equation does not have only one solution; it has an entire *family* of solutions. The family of solutions is called the *general solution* of the differential equation.

The problem of finding a particular solution satisfying certain initial conditions (for example, at time $t = 0$, the quantity is $Q = 5$) is called an *initial value problem.*

Exercises

4. Consider the differential equation $y' = 2x$.

   (a) What is the general solution of this differential equation?

   (b) Sketch three distinct particular solutions of the differential equation.

   (c) Find a formula for the particular solution satisfying $y(0) = -4$.

5. Solve the initial value problem: $\frac{dy}{dx} = x\cos(x^2)$, $y(0) = 1$.

4. Verifying solutions

In general, it is difficult to find formulas for solutions of differential equations, but if we are given a purported solution (or family of solutions) it is easy to *check* whether or not it really is a solution: simply "plug in" the function to the left side of the differential equation and to the right side of the differential equation and see if you get the same thing. The key is to keep the two sides of the equation separate while you are simplifying them. See Example 1.

Exercises

6. Consider the differential equation $x^2\,y' + xy = 1$.

   (a) Show that every member of the family $y = (\ln(x) + C)/x$ is a solution of this differential equation. (See Example 1.)

   (b) Illustrate by graphing several members of the family of solutions on a common screen. (Use *Mathematica* or a graphing calculator.) Sketch your results below.

   (c) Find a particular solution that satisfies the initial condition $y(1) = 2$. (See Example 2.)

(d) Find a particular solution that satisfies the initial condition $y(2) = 1$.

7. Consider the differential equation $y' = xy^3$.

   (a) What can you say about the graph of a solution when x is close to 0?

   What if x is large?

   (b) Verify that all members of the family $y = (c - x^2)^{-1/2}$, with $c > 0$, are solutions of the differential equation.

   (c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?

   (d) Find a solution of the initial-value problem:

$$y' = xy^3 \qquad y(0) = 2$$

8. Match solutions and differential equations. (Note: Each equation may have more than one solution, or no solutions.)

   Differential Equations:

   (a) $y' = y/x$ (b) $y' = 3y/x$ (c) $y' = 3x$ (d) $y' = y$ (e) $y' = 3y$

   Purported Solutions:

   (i) $y = x^3$ (ii) $y = 3x$ (iii) $y = e^{3x}$ (iv) $y = 3e^x$ (v) $y = x$