**Name:** ______________________ **Section:** ________

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## Main Points:

1. The Law of Natural Growth
2. Emigration or "harvesting" at a constant rate
3. The Logistic Model

## 1. The Law of Natural Growth

One of the simplest models of population growth is the Law of Natural Growth, which states that a population grows at a rate proportional to its size:

$$\tfrac{dP}{dt} = kP\,, \qquad (k > 0)\,.$$

**Exercises.**

1. For what values of $P$ is $P(t)$ constant? Increasing? Decreasing?

2. Sketch three solution curves that have different shapes. Which curve(s) has/have practical meaning in terms of population size?

We may obtain a formula for the general solution to this differential equation using a technique knows as separation of variables. (See the discussion in the textbook, which occurs between Equation 1 and Equation 2.) The general solution is:

$$P(t) = Ae^{kt}, \text{ where } A \text{ is any real number.}$$

Note that there is a natural interpretation of the constant $A$ in this case; it is the initial value $A = P(0)$.

## 2. Emigration or "harvesting" at a constant rate

If a population is also losing members due to emigration or harvesting at a constant rate $m$, we may model the rate of change of the population by the differential equation:

$$\frac{dP}{dt} = kP - m\,.$$

**Exercises.**

3. For what values of $P$ does $P$ increase? Decrease? Stay constant?

4. Sketch three solution curves that have different shapes. Which curve(s) has/have practical meaning in terms of population size?

5. In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population (so the relative growth rate was $k = 0.016$). Because of the potato famine in the 1840s and 50s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

6. Imitate the method used to derive the general solution to the differential equation $\frac{dP}{dt}$ to derive the general solution to the differential equation $\frac{dP}{dt} = kP - m$.

## 3. The Logistic Model

Although the Law of Natural Growth is reasonable for a population in early stages of growth, often there is a natural limit to the size of a population. For example, a herd of elephants living in a certain geographic region cannot increase in size indefinitely, since at a certain point the land cannot provide enough food for the herd if it keeps increasing in size. In general, the maximum population, $M$, that an environment is capable of sustaining in the long run is called the **carrying capacity**, and the differential equation that takes this into account is the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

Notice that this may be written in the form:

$$\frac{dP}{dt} = (\text{const.}) \cdot P \cdot (M - P)\ ,$$

where the constant is $k/M$, which is positive. Thus we see that $\frac{dP}{dt}$ is jointly proportional to $P$ and to the difference between $M$ and $P$.

**Exercises.**

7. For what values of $P$ does $P$ increase? Decrease? Stay constant?

8. For each equilibrium solution found above, determine whether the equilibrium is stable or unstable. (It is stable if a small increase or decrease in $P$ results in a return to the equilibrium.)

9. Sketch three non-constant solution curves that have different shapes. Which curve(s) has/have practical meaning in terms of population size?

The general solution to the logistic equation may be derived in a manner similar to the other differential equations for population growth discussed above; in the integration step, the key technique is the Partial Fractions Decomposition. The general solution is derived in Example 1 in the textbook. It is:

$$P(t) \;=\; \frac{M}{1+Ae^{-kt}}\;, \quad \text{where} \quad A \;=\; \frac{M-P_0}{P_0}\;.$$

**Exercises.**

10. A population grows according to the following logistic equation, where $t$ is measured in weeks:

$$\frac{dP}{dt} \;=\; 0.04P\left(1-\frac{P}{1200}\right), \qquad P(0)=60\;.$$

(a) What is the carrying capacity? What is the value of $k$?

(b) Write the solution of the equation.

(c) What is the population after 10 weeks?