Math 132, Prep 17.4 (Series Solutions for Differential Equations)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. Not every differential equation has an elementary solution.
2. Assume that a power series solution exists and attempt to find the coefficients.
3. Use the DE to derive a "recurrence relation" for the coefficients.

Recall that not every elementary function has an elementary antiderivative. For example $f(x) = e^{-x^2}$ is elementary, and we may construct an antiderivative using the FTC: $F(x) = \int_0^x e^{-t^2}\,dt$, but this function *cannot* be written as a combination of functions that are familiar from precalculus. This function is important, because it is commonly used to compute probabilities. We have learned two ways of answering questions about non-elementary antiderivatives: numerical integration and power series. Both work for definite integrals; power series are useful for indefinite integrals as well.

More generally, not every differential equation has an elementary solution. For example, the differential equation,

$$y'' - 2xy' = 0\,,$$

which arises in connection with the Schrödinger equation in quantum mechanics, does not have an elementary solution.

Our strategy will be to *suppose* that a solution $y = f(x)$ exists and that it can be represented by a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots .$$

Then we use the differential equation along with algebra to determine what the coefficients must be.

Exercises.

1. Consider the differential equation $y' - y = 0$.

   (a) We suppose that this DE has a solution $y = f(x)$, and this function has a power series representation $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Find a power series representation for $y' = f'(x)$.

   (b) Use the power series for $f(x)$ and the power series for $f'(x)$ to find a power series for $f'(x) - f(x)$.

(c) Since we are assuming that $y = f(x)$ is a solution to the DE, the power series for $f'(x) - f(x)$ must be equal to zero for all x. This means all its coefficients must be zero. Use this fact to derive a "recurrence relation" for the c_n's; in this case it will be an equation relating c_{n+1} and c_n.

(d) We will not try to find a value for c_0. (Since we are solving a first order DE, there will be one constant that we cannot eliminate.) But we hope to get all the other coefficients in terms of c_0. Start by solving for c_1 in terms of c_0, using the recurrence relation.

(e) Now solve for c_2 in terms of c_1, and use your previous work to find c_2 in terms of c_0.

(f) Continue in this way until you can guess a formula for c_n in terms of c_0.

(g) Now write out the power series for $f(x)$, in terms of c_0.

(h) Check that this is a solution to the differential equation, and find the interval of convergence. Do you recognize this as an elementary function?

2. Use the method outlined in the previous problem to find a series solution to the differential equation:

$$y'' + xy' + +y = 0 \,.$$

Notes:

- This differential equation is of **order two** since it has a second-order derivative (y'') in it.
- You need to find power series for two derivatives of your supposed solution instead of just one.
- The recurrence relation will be of **depth two**, meaning that c_{n+2} will be related to c_{n+1} and c_n.
- There will be two constants in the final answer; choose c_0 and c_1 to remain.

3. Find a series solution to $(x-1)y'' + y' = 0$.

4. Consider the initial value problem:

$$x^2y'' + xy' + x^2y = 0\,; \qquad y(0) = 1\,,\;\; y'(0) = 0\,.$$

The solution is called a Bessel function of order zero.

(a) Solve the IVP to get a series representation for the Bessel function.

(b) Graph several Taylor polynomials until you get a good approximation for the Bessel function on the window $[-5, 5]$.