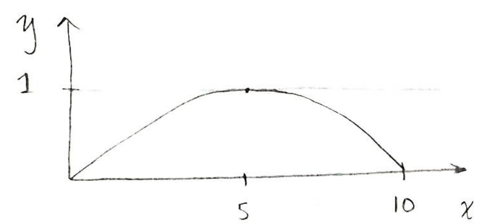
Practice w/ Concepts : Exam 1

1) Vibrating guitar string: $f(x,t) = \cos t \, \sin(\pi x/10)$, $\quad 0 \le x \le 10$

a) $f(5,0) = \cos(0)\sin(\pi/2) = 1$

At the moment the string is plucked, the vertical displacement of the midpoint of the string is 1 unit.

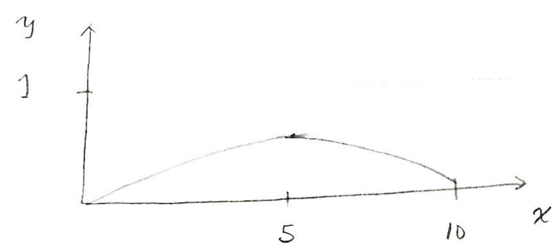
b) $f(x,0)$: The vertical displacent of a point on the string x units from the end at the moment the string is plucked.



$f(x,0) = \sin(\pi x/10)$

(Time fixed, horiz. position along string varies)

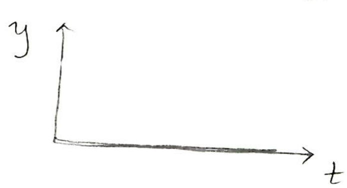
$f(x,1)$: The vert. displ. of a pt on the string x units from the end, 1 unit of time after the string has been plucked (Time fixed, horiz. position along string varies)



$f(x,1) = \cos(1)\sin(\pi x/10)$

$\approx 0.54 \sin(\pi x/10)$

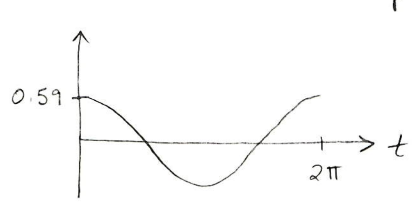
c) $f(0,t)$: The vert disp. of the end of the string as it dep. on time.



$f(0,t) = 0$

The end of the string always has a vertical displacement of zero; it does not move as time goes on.

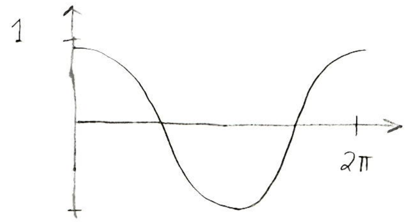
$f(2,t)$: The vert. disp. of a pt 2 units from end of string as it dep on time



$f(2,t) = \sin(\pi/5)\cos t \approx 0.59 \cos t$

1c) con't

$f(4,t)$: The vert. displacement of a pt on the string 4 units from the end, as it dep on time.



$f(4,t) = \sin(2\pi/5)\cos t \approx 0.95 \cos t$

d) $f_t(x,t) = -\sin(t)\sin(\pi x/10)$

$f_t(5,0) = -\sin(0)\sin(\pi/2) = 0$ (initial vert. vel. at $x=5$)

Initially, the midpoint of the string is not moving.

$f_t(5,\pi/2) = -\sin(\pi/2)\sin(\pi/2) = -1$

After $\pi/2$ units of time, the midpoint of the string is moving downwards w/ a speed of one unit per unit time.

$f_t(5,\pi) = -\sin(\pi)\sin(\pi/2) = 0$

After π units of time the midpoint of the string is momentarily paused. (This is when it has reached its maximum downward displacement, and it is about to start moving upwards.)

e) $f_x(x,t) = \cos t \cdot (\frac{\pi}{10})\cos(\pi x/10) = (\pi/10)\cos t \cos(\pi x/10)$

$f_x(0,0) = (\pi/10)\cos(0)\cos(0) = \pi/10 \approx 0.314$

Initially, the slope of the string at the left end is 0.314.

$f_x(2,0) = (\pi/10)\cos(0)\cos(\pi/5) \approx 0.25$

Initially, the slope of the string at a point 2 units from the left end is about 0.25.

1e) con't

$f_x(5,0) = (\pi/10)\cos(0)\cos(\pi/2) = 0$

Initially, the slope of the string at its midpoint is zero, i.e. the tangent would be horizontal.

f.) $f_{xx}(x,t) = (\pi/10)\cos t \cdot (\pi/10)(-\sin(\pi x/10)) = -\pi^2/100 \cos t \sin(\pi x/10)$

$f_{xx}(4,0) = -\pi^2/100 \cos(0)\sin(2\pi/5) \approx -0.094$

Initially, the string is concave down at a point 4 units from the end. In particular, for a small positive displacement Δx from the position $x=4$, the slope of the string will decrease by about $0.094\,\Delta x$.

$f_{xt}(x,t) = \frac{\partial}{\partial t}(f_x(x,t)) = -\pi/10 \sin t \cos(\pi x/10)$

$f_{xt}(4,0) = -\pi/10 \sin(0)\cos(2\pi/5) \approx -0.097$

Initially, the slope of the string at a point 4 units from the end is decreasing at a rate of about 0.097 per unit time.

$f_{tx}(x,t) = \frac{\partial}{\partial x}(f_t(x,t)) = -\pi/10 \sin(t)\cos(\pi x/10)$

$f_{tx}(4,0) \approx -0.097$

Initially, the vertical velocities of the points on the string that are about 4 units away from the left end of the string decrease from left to right. In particular, for a small positive displacement Δx from the position $x=4$, the velocity decreases by about $0.097\,\Delta x$.

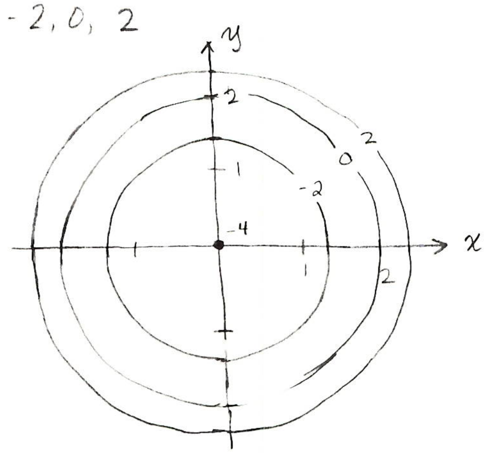
3b) $f(x,y) = x^2 + y^2 - 4$: z-values : $-4, -2, 0, 2$

$0 = x^2 + y^2 - 4 \iff x^2 + y^2 = 4$

$-2 = x^2 + y^2 - 4 \iff x^2 + y^2 = 2$

$-4 = x^2 + y^2 - 4 \iff x^2 + y^2 = 0$

$2 = x^2 + y^2 - 4 \iff x^2 + y^2 = 6$



c) $F(x,y,z) = 2$:

$$z + 4 = x^2 + y^2$$
$$4 = x^2 + y^2 - z$$
$$2 = \underbrace{x^2 + y^2 - z - 2}_{F(x,y,z)}$$

$F(x,y,z) = x^2 + y^2 - z - 2$

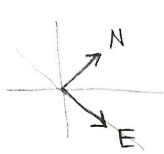
d) $\vec{\nabla} f(x,y) = 2x\hat{\imath} + 2y\hat{\jmath}$

$\vec{\nabla} f(1,1) = 2\hat{\imath} + 2\hat{\jmath}$ points in dir of steepest ascent, so points north

Perhaps simpler to say $\hat{\imath} + \hat{\jmath}$ points due north.

e) Slope of path going north : $\| \vec{\nabla} f(1,1) \| = \sqrt{4+4} = 2\sqrt{2}$ (b/c steepest ascent)

Slope of path going east:



perpendicular to gradient, so along contour line

$\Rightarrow$ slope is zero

Slope of path in dir of $\vec{v} = \hat{\imath} + 3\hat{\jmath}$: unit vector $\hat{v} = \frac{1}{\sqrt{10}}\hat{\imath} + \frac{3}{\sqrt{10}}\hat{\jmath}$

$f_{\hat{v}}(1,1) = \vec{\nabla} f(1,1) \cdot \hat{v} = \frac{2}{\sqrt{10}} + \frac{6}{\sqrt{10}} = \frac{8}{\sqrt{10}} \approx 2.52$ is slope in dir of $\vec{v}$.

3f.) Tangent plane at $(1,1,-2)$.

$$m = f_x(1,1) = 2$$

$$n = f_y(1,1) = 2$$

$$z + 2 = 2(x-1) + 2(y-1)$$

g.) Tangent plane at $(2,0,0)$

$$\vec{n} = \vec{\nabla} F(2,0,0) \quad : \quad \vec{\nabla} F = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$$

$$= 4\hat{\imath} - \hat{k}$$

$$4(x-2) + 0(y-0) - 1(z-0) = 0$$

$$4(x-2) - z = 0$$

4.) $\vec{v} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$, $\vec{w} = 4\hat{\imath} - \hat{\jmath} + 2\hat{k}$

a) Parallel to $\vec{v}$: $2\vec{v} = 4\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$

$3\vec{v} = 6\hat{\imath} + 9\hat{\jmath} + 3\hat{k}$ (& any other scalar mult.)

Perpendicular to $\vec{v}$: $\vec{v} \cdot \vec{n} = 0$

$$(2\hat{\imath} + 3\hat{\jmath} + \hat{k}) \cdot (a\hat{\imath} + b\hat{\jmath} + c\hat{k}) = 0$$

$$2a + 3b + c = 0$$

Choose $a = 3,\ b = -2,\ c = 0$: $3\hat{\imath} - 2\hat{\jmath}$ (& many others)

Choose $a = 0,\ b = 1,\ c = -3$: $\hat{\jmath} - 3\hat{k}$

b) $\vec{w}$ is clearly not a scalar multiple of $\vec{v}$ since scalar mults of $\vec{v}$ are of the form : $2t\hat{\imath} + 3t\hat{\jmath} + t\hat{k}$ for some real number t. Looking at the $\hat{\imath}$ component of $\vec{w}$, we would need $t = 2$, but this does not work for the $\hat{\jmath}$ or $\hat{k}$ component.

4b) con't

If $\vec{v} \perp \vec{w}$, then $\vec{v} \cdot \vec{w} = 0$ but

$$\vec{v} \cdot \vec{w} = 8 - 3 + 2 = 7 \neq 0$$

So $\vec{v}$ & $\vec{w}$ are not $\perp$.

c) $$\vec{v}_{parallel} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} \right) \vec{w} = \frac{7}{\sqrt{16+1+4}} \vec{w} = \frac{7}{\sqrt{21}} \vec{w} = \frac{28}{\sqrt{21}} \hat{\imath} - \frac{1}{\sqrt{21}} \hat{\jmath} + \frac{2}{\sqrt{21}} \hat{k}$$

$$\vec{v}_{perp} = \vec{v} - \vec{v}_{parallel} = \left(\frac{2\sqrt{21}}{\sqrt{21}} - \frac{28}{\sqrt{21}} \right) \hat{\imath} + \left(\frac{3\sqrt{21}}{\sqrt{21}} - \frac{1}{\sqrt{21}} \right) \hat{\jmath} + \left(\frac{\sqrt{21}}{\sqrt{21}} - \frac{2}{\sqrt{21}} \right) \hat{k}$$

d) $$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 4 & -1 & 2 \end{vmatrix} = (6+1)\hat{\imath} + (4-4)\hat{\jmath} + (-2-12)\hat{k} = 7\hat{\imath} - 14\hat{k}$$

$\vec{n}_1 = 7\hat{\imath} - 14\hat{k}$

$\vec{n}_2 = -7\hat{\imath} + 14\hat{k}$

$\vec{n}_3 = \hat{\imath} - 2\hat{k}$

could choose any nonzero scalar multiples of $\vec{v} \times \vec{w}$

e) $$A = \|\vec{v} \times \vec{w}\| = \sqrt{7^2 + 14^2} = 7\sqrt{1^2 + 2^2} = 7\sqrt{5}$$