Optimization : Method of Lagrange Multipliers

Context ("Constrained" Optimization)

- Problem : Find max/min vals. for function on restricted domain.
- Strategy :
  1) List of candidate points
  2) Plug in to function
  3) Compare & conclude.
- Applications :
  - "objective" function f
  - constraint condition : $g \le c$ or $g = c$

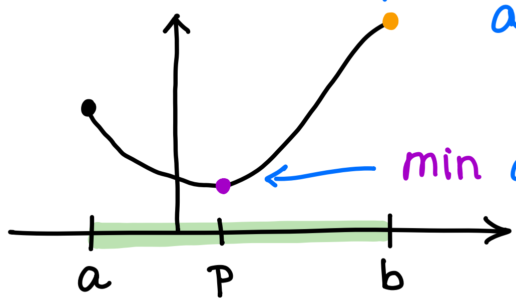
Key Ideas

- A ________ function on a ____, ______ domain achieves a max val & a min val.
- Max/min vals achieved either at ____ in ______ of domain or somewhere on ______ of domain.

Single Var :

$f(x)$ on $[a,b]$

CP $x=p$ in (a,b)



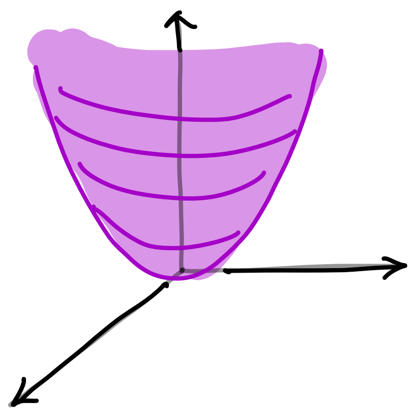
max achieved at boundary of domain

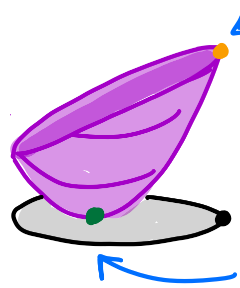
min achieved at C.P. in interior of domain

Multi-Var :

$f(x,y) = x^2+y^2$

- domain : whole plane
- global min @ C.P.
- no max val.



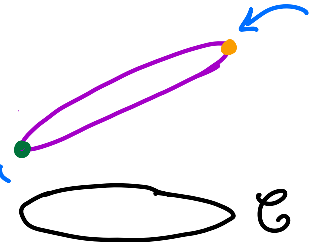


max achieved at boundary of domain

min achieved at C.P. in interior of domain

Restrict domain to region in xy-plane

Restrict domain to curve in xy-plane.



max achieved at pt. on boundary of domain

min achieved at pt on boundary of domain

Lagrange Multipliers (Concepts)

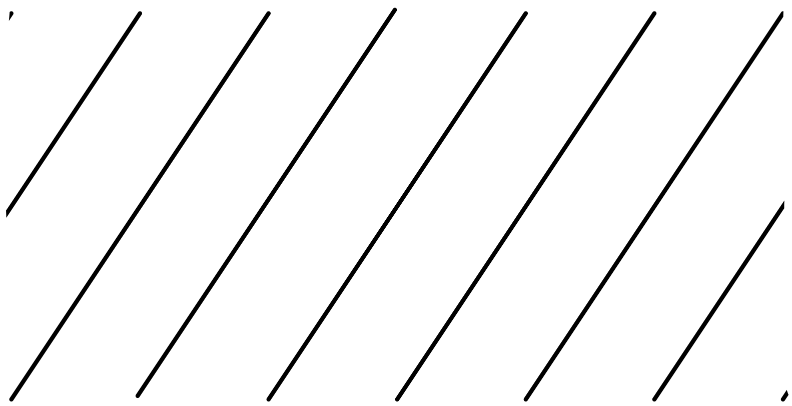
Find max/min vals. of $f(x,y) = 3x - 2y$ subject to the constraint $x^2 + 2y^2 = 44$.

Objective Function :

Constraint :

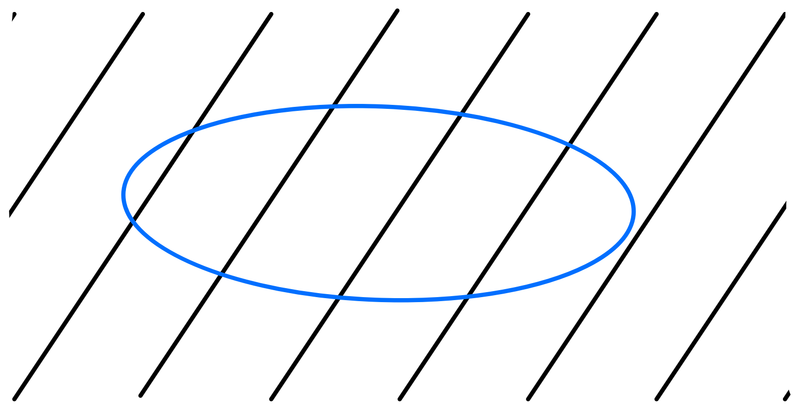
Contour diagram for $f(x,y) = 3x - 2y$

$$c = 3x - 2y \Rightarrow y = \frac{1}{2}(3x - c)$$

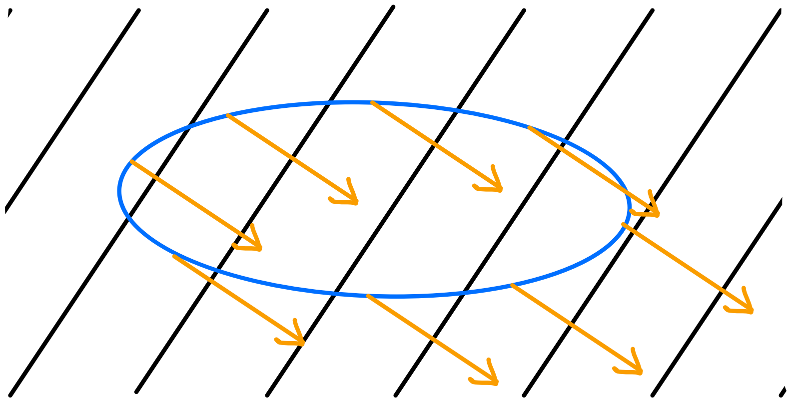


Note :
$\vec{\nabla} f =$

Restrict domain to curve : $x^2 + 2y^2 = 44$



Where will the max value be? Min val?



Notice: max/min occur when

$\vec{\nabla} f \perp$ ________________

$\vec{\nabla} f \parallel$ ________________

Introduce auxilliary function $g(x,y)$ for which the constraint curve is a contour.

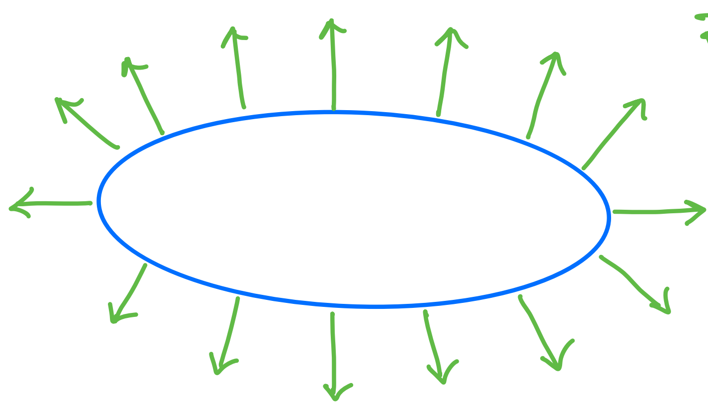
Constraint: $x^2 + 2y^2 = 44$

Let $g(x,y) =$ ____________

Then constraint curve is the $z =$ ____ contour for $g(x,y)$.

Then, by design,

$\vec{\nabla} g \perp$ ________________

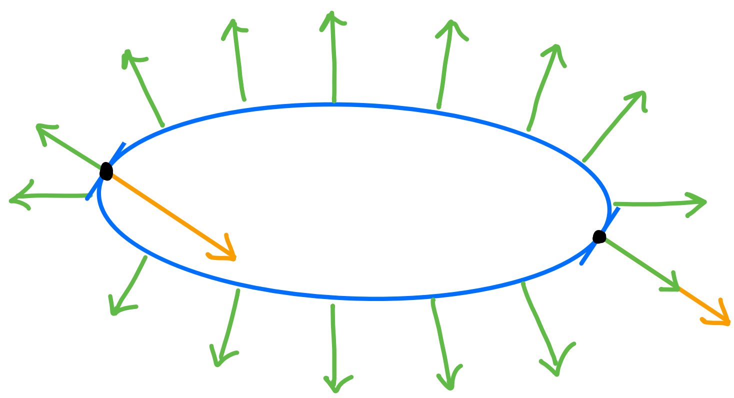


$\vec{\nabla} g = 2x\,\hat{\imath} + 4y\,\hat{\jmath}$

Find points on ellipse where

$\vec{\nabla} f \parallel \vec{\nabla} g$ i.e. $\vec{\nabla} f = (\text{scalar})\, \vec{\nabla} g$

call this "λ" (lambda)



Computations:

- $\vec{\nabla} f = 3\hat{\imath} - 2\hat{\jmath}$ $\qquad \vec{\nabla} g = 2x\,\hat{\imath} + 4y\,\hat{\jmath}$
- $\vec{\nabla} f = \lambda \vec{\nabla} g$ means:

$$\begin{cases} \square = \lambda\, \square & (1) \\ \square = \lambda\, \square & (2) \end{cases}$$

- Want points on the constraint curve

$$\begin{cases} 3 = 2\lambda x & (1) \\ -2 = 4\lambda y & (2) \\ & (3) \end{cases}$$

- 3 equations & 3 unknowns, but we don't care what λ is... just want x & y.
  - often try to eliminate ____
  - careful not to ____________

$$\begin{cases} 3 = 2\lambda x & (1) \\ -2 = 4\lambda y & (2) \\ x^2 + 2y^2 = 44 & (3) \end{cases}$$

$(1) \Rightarrow \lambda = \frac{3}{2x}$ (since $x \neq 0$, b/c $3 \neq 0$)

$\downarrow$

$(2)\ -2 = 4\lambda y$

- Two candidate points: ______ & ______
- Evaluate f:

$$f(x,y) = 3x - 2y$$

$$f(\quad,\quad) =$$

$$f(\quad,\quad) =$$

- Compare & Conclude

For $f(x,y)$ restricted to curve $x^2+2y^2=44$,

Global max val.: ______ at ______

Global min val: ______ at ______

Method of Lagrange Multipliers

When optimizing objective function f, subject to constraint of the form $g=c$ or $g \leq c$, use this method to get list of candidate points on the curve $g=c$.

Note If constraint is of the form $g \leq c$, (i.e. we are restricting to a region in the xy-plane), we must also consider critical points in the interior $(g<c)$ as candidates.

Candidate Points On Curve $g=c$ are

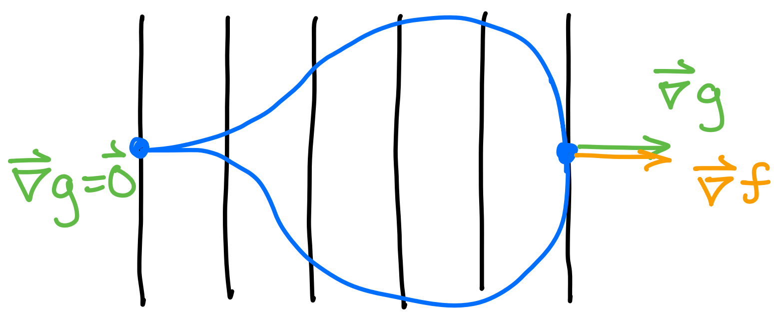
points where:

- $\vec{\nabla} f = \lambda \vec{\nabla} g$ for some real number λ and $g=c$ } L.M.

- $\vec{\nabla} g = \vec{0}$ and $g=c$
- curve has endpoints (if any) } L.M. does not find

Why include points where $\vec{\nabla} g = \vec{0}$?

<u>Ex</u> $f(x,y) = x$; Constraint: $y^2 + x^4 - x^3 = 0$

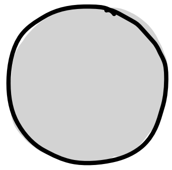


Example Optimize $f(x,y) = x^2 + 2y^2$, subject to the constraint $x^2 + y^2 \leq 4$.

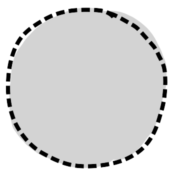
Objective Function: $f(x,y) = x^2 + 2y^2$

Domain: $x^2 + y^2 \leq 4$

Introduce g: $g(x,y) = x^2 + y^2$



Interior



$x^2 + y^2 < 4$

Critical Points:

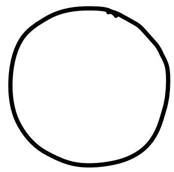
- $\vec{\nabla} f(P) = \vec{0}$

or

- $\vec{\nabla} f(P)$ DNE

(and $g(P) < 4$)

Boundary



$x^2 + y^2 = 4$

Points where:

- $\vec{\nabla} f = \lambda \vec{\nabla} g$ (L.M.)

or

- $\vec{\nabla} g = \vec{0}$

or

- endpoints

(and $g(P) = 4$)

① Find critical points in the interior.

$$f(x,y) = x^2 + 2y^2$$

$$\vec{\nabla} f(x,y) = \_\_\_\,\hat{\imath} + \_\_\_\,\hat{\jmath}$$

$= \vec{0}$

DNE?

$$\begin{cases} \quad = 0 & (1) \\ \quad = 0 & (2) \end{cases}$$

Check : C.P. in interior $x^2+y^2 < 4$?

② Additional Candidate Points on Boundary

$$g(x,y) = x^2 + y^2$$

(a) $\vec{\nabla} g(x,y) = \vec{0}$? (& on boundary)

$$\vec{\nabla} g(x,y) = \_\_\_\,\hat{\imath} + \_\_\_\,\hat{\jmath}$$

$$\begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_ & (1) \\ \_\_\_\_\_\_\_\_\_\_\_\_ & (2) \\ \_\_\_\_\_\_\_\_\_\_\_\_ & (3) \end{cases}$$

(b) Endpoints of boundary curve?

(c) $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$ (& on boundary)

$$\vec{\nabla} f(x,y) = 2x\,\hat{\imath} + 4y\,\hat{\jmath}$$

$$\vec{\nabla} g(x,y) = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$$

$$\begin{cases} 2x = 2x\lambda & (1) \\ 4y = 2y\lambda & (2) \\ x^2 + y^2 = 4 & (3) \end{cases}$$

List of Candidate Points

③ Evaluate f at candidate points

$f(x,y) = x^2 + 2y^2$

$f(0,0) =$

$f(2,0) =$

$f(-2,0) =$

$f(0,2) =$

$f(0,-2) =$

⑤ Compare & Conclude

Max val : ______ at ____________________

Min val : ______ at ____________________

Example $f(x,y) = x + 3y$; $x^2 + y^2 \leq 2$

Let $g(x,y) =$

① Crit. Pts. in Interior

$\vec{\nabla} f(x,y) = \_\_\_\_\_\ \hat{\imath} + \_\_\_\_\_\ \hat{\jmath}$

$\vec{\nabla} f = \vec{0}$? $\vec{\nabla} f$ DNE ?

② Additional Candidate Points on Boundary

(a) $\vec{\nabla} g = \vec{0}$ & $g = 2$

$g(x,y) = x^2 + y^2$

$\vec{\nabla} g(x,y) = \_\_\_\_ \hat{i} + \_\_\_\_ \hat{j}$

$$\begin{cases} & (1) \\ & (2) \\ & (3) \end{cases}$$

(b) $\vec{\nabla} f = \lambda \vec{\nabla} g$ & $g = 2$

$\vec{\nabla} f = \hat{i} + 3\hat{j}$; $\vec{\nabla} g = 2x\hat{i} + 2y\hat{j}$

$$\begin{cases} 1 = 2x\lambda & (1) \\ 3 = 2y\lambda & (2) \\ x^2 + y^2 = 2 & (3) \end{cases}$$