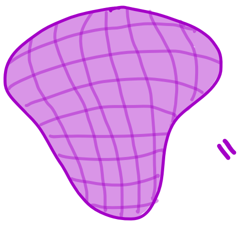
The Divergence Theorem

<u>Thm</u> Given W, a solid region with pws boundary S (surface) & $\vec{F}$, a smooth vf on an open domain containing W & S,

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} \, dV$$

oriented outward



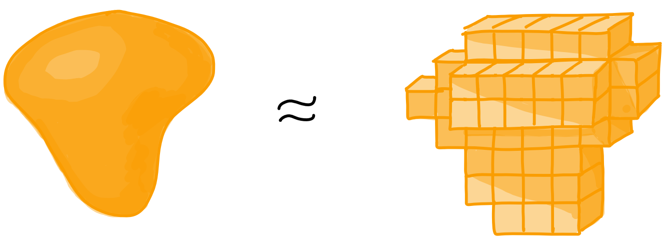
"skin"

S

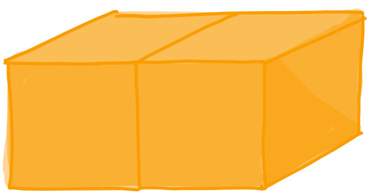
"flesh"

W

Break W into small regions & approximate regions with rectangular blocks: $W_1, W_2, \ldots, W_n$

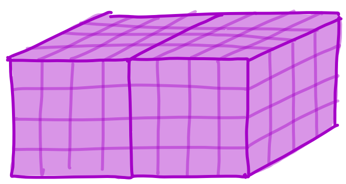


solid blocks



$\ldots$ W_i W_{i+1} $\ldots$

boundary surfaces: boxes

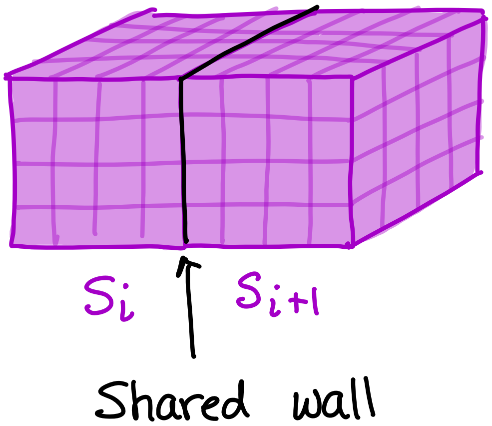


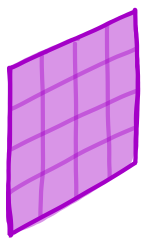
$\ldots$ S_i S_{i+1} $\ldots$

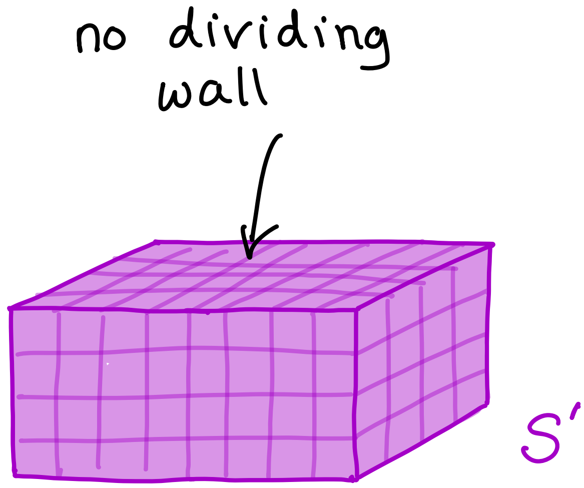
Since $\operatorname{div}\vec{F}$ is flux density,

$$\operatorname{div}\vec{F} \approx \left(\begin{matrix}\text{Flux out}\\ \text{of } S_i\end{matrix}\right) \Big/ \left(\begin{matrix}\text{vol.}\\ \text{of } W_i\end{matrix}\right) \leftarrow \Delta V$$

$$\Rightarrow \left(\begin{matrix}\text{Flux out}\\ \text{of } S_i\end{matrix}\right) \approx \operatorname{div}\vec{F}\ \Delta V$$

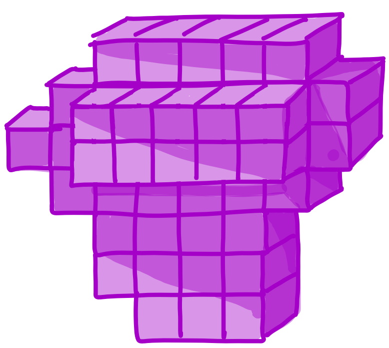






$$\left(\begin{matrix}\text{Flux out}\\ \text{of } S_i\end{matrix}\right) + \left(\begin{matrix}\text{Flux out}\\ \text{of } S_{i+1}\end{matrix}\right) = \left(\begin{matrix}\text{Flux out}\\ \text{of } S'\end{matrix}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \left(\begin{matrix}\text{Flux out}\\ \text{of } S_i\end{matrix}\right) \approx \left(\begin{matrix}\text{Flux out}\\ \text{of } S\end{matrix}\right) = \int_S \vec{F}\cdot d\vec{A}$$



On the other hand,

$$\sum_{i=1}^{n} \left(\begin{matrix}\text{Flux out} \\ \text{of } S_i\end{matrix} \right) \approx \sum_{i=1}^{n} (\text{div}\, \vec{F})(\Delta V)$$

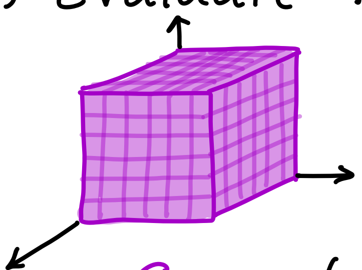
$$\Big\downarrow \; n \to \infty \quad (\Delta V \to 0)$$

$$\begin{matrix}\text{Flux out} \\ \text{of } S\end{matrix} = \int_S \vec{F} \cdot d\vec{A} = \boxed{\qquad\qquad\qquad}$$

<u>Example</u> $\vec{F} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Compute flux in 2 ways:

(1) Evaluate flux integral directly



S

$$\int_S \vec{F} \cdot d\vec{A}$$

(calc. flux through each face & add)

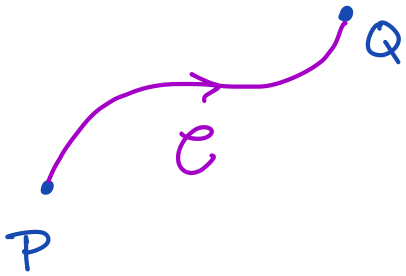
(2) Use Divergence Thm.

div $\vec{F} = \ldots$

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \text{div}\, \vec{F}\, dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, dx\,dy\,dz$$

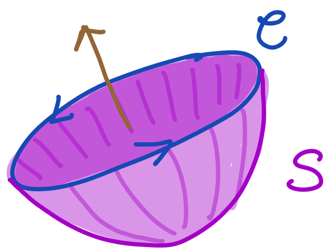
Three Fundamental Theorems

(1) FTC for Line Integrals (FTCLI)



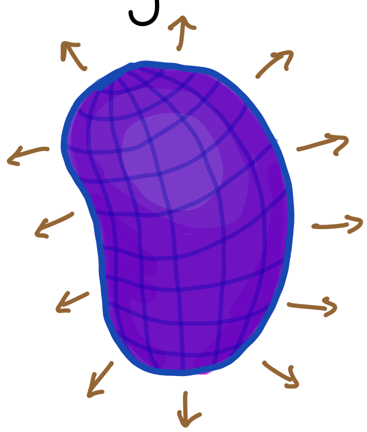
$$\int_{\mathcal{C}} \operatorname{grad} f \cdot d\vec{r} = f(Q) - f(P)$$

(2) Stokes' Theorem

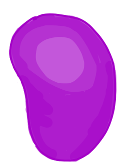


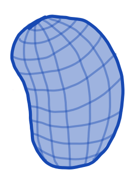
$$\int_S \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

(3) Divergence Theorem



$$\int_W \operatorname{div} \vec{F} \, dV = \int_S \vec{F} \cdot d\vec{A}$$





<u>Pattern</u> $\int_{\text{n-dim'l object}} \left(\text{deriv. type object}\right) = \int_{\text{boundary of n-dim'l object}} \left(\quad \right)$

Methods for Computing Line & Flux Integrals

Line Integrals

① Parameterization

② FTCLI :

- Need vf to be ________
- Check using _____
- If domain has no holes & curl = 0, find ______________ & plug in endpoints of path.

③ Green's Thm

- vf is ___ dimensional
- curve is _______
- curve & enclosed region in _______ of vf
- compute ____ & integrate over region

④ Stokes' Thm

- vf is ___ dimensional
- curve is ______
- curve & "spanning surface" in _______ of vf
- compute (3D vector) curl & calculate flux through surface

Flux & Flux Integrals

① Basic : vf ________ & surface ________

- Dot product

② Compute flux integral directly

- $d\vec{A} = \hat{n}\, dA$, ($\vec{F}$ on S), dot prod., ...

③ Divergence Thm

- surface must be ________
- surface & enclosed solid in ________
- calculate divergence & integrate over solid region enclosed by surface

④ Stokes' Thm

- vf must be a ______ field
- convert flux integral to
  - line int. along boundary → must find ________

  OR
  - flux integral through simpler surface