

# The Divergence Theorem

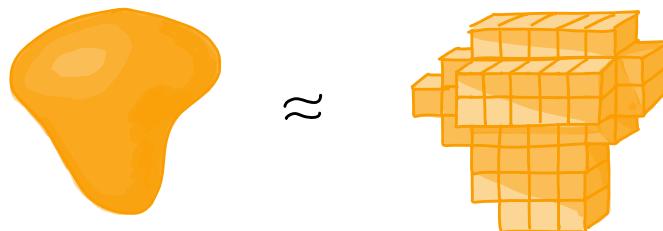
Thm Given  $W$ , a solid region with pws boundary  $S$  (surface) &  $\vec{F}$ , a smooth vf on an open domain containing  $W \& S$ ,

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} dV$$

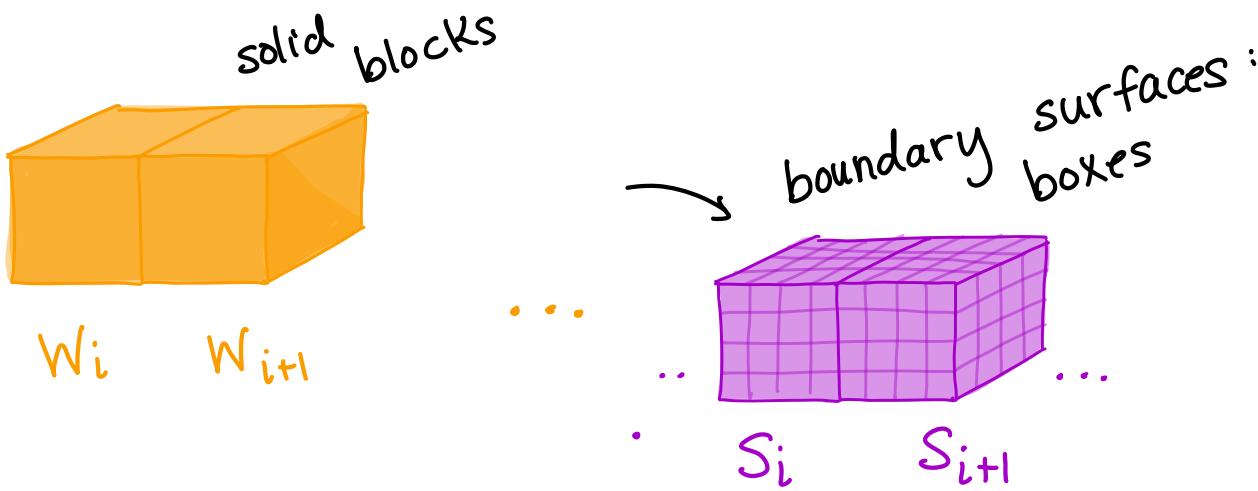
↑  
oriented outward



Break  $W$  into small regions & approximate regions with rectangular blocks:  $W_1, W_2, \dots, W_n$



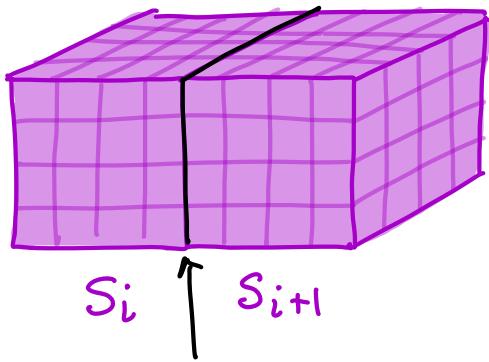
$\approx$



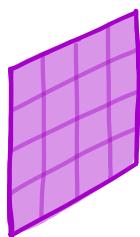
Since  $\operatorname{div} \vec{F}$  is flux density,

$$\operatorname{div} \vec{F} \approx \left( \frac{\text{Flux out}}{\text{of } S_i} \right) / \left( \frac{\text{vol.}}{\text{of } W_i} \right)$$

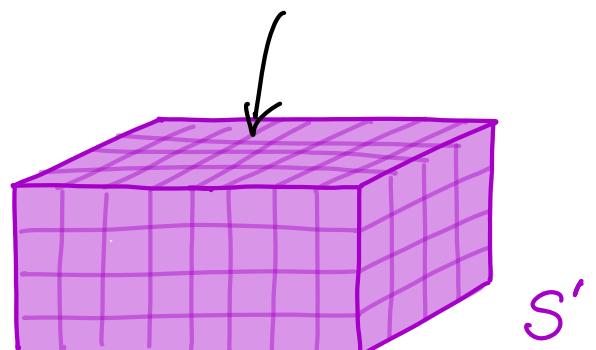
$$\Rightarrow \left( \frac{\text{Flux out}}{\text{of } S_i} \right) \approx \operatorname{div} \vec{F} \Delta V$$



Shared wall



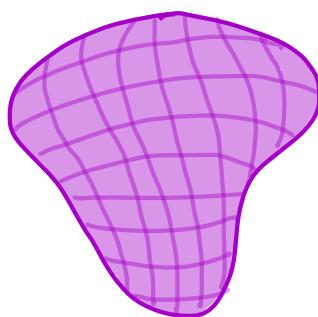
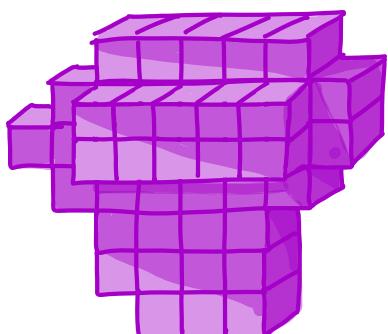
no dividing wall



$S'$

$$\text{Flux out of } S_i + \text{Flux out of } S_{i+1} = \text{Flux out of } S'$$

$$\Rightarrow \sum_{i=1}^n \left( \frac{\text{Flux out}}{\text{of } S_i} \right) \approx \text{Flux out of } S = \int_S \vec{F} \cdot d\vec{A}$$

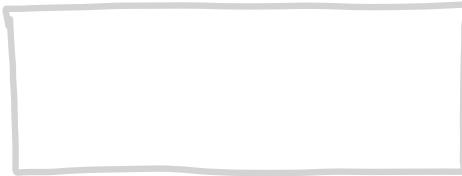


On the other hand,

$$\sum_{i=1}^n (\text{Flux out of } S_i) \approx \sum_{i=1}^n (\operatorname{div} \vec{F})(\Delta V)$$

$$\downarrow n \rightarrow \infty (\Delta V \rightarrow 0)$$

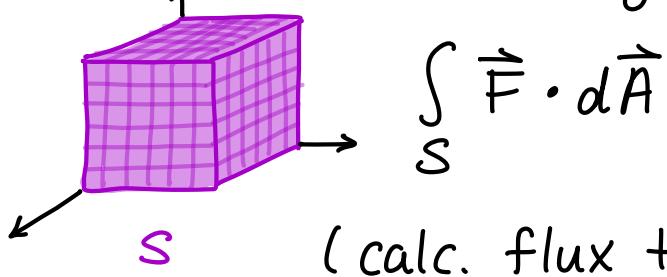
$$\text{Flux out of } S = \int_S \vec{F} \cdot d\vec{A} =$$



Example  $\vec{F} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Compute flux in 2 ways:

(1) Evaluate flux integral directly



(calc. flux through each face & add)

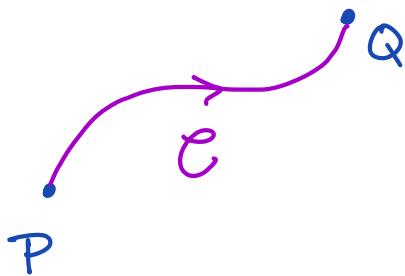
(2) Use Divergence Thm.

$$\operatorname{div} \vec{F} = \dots$$

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} dV = \iiint \underline{\quad} dx dy dz$$

# Three Fundamental Theorems

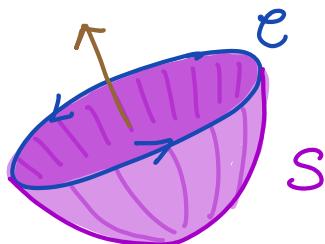
(1) FTC for Line Integrals (FTCL1)



A hand-drawn diagram of a wavy purple line representing a curve  $C$ . The curve starts at a blue dot labeled  $P$  and ends at a blue dot labeled  $Q$ .

$$\int_C \text{grad } f \cdot d\vec{r} = f(Q) - f(P)$$

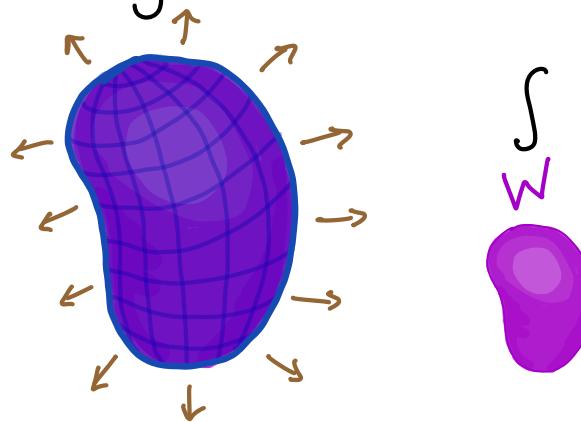
(2) Stokes' Theorem



A hand-drawn diagram of a surface  $S$  shaded in purple. A blue line representing the boundary  $C$  is drawn on the surface. An arrow points along the curve  $C$ , indicating its orientation.

$$\int_S \text{curl } \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}$$

(3) Divergence Theorem



A hand-drawn diagram of a volume  $W$  shaded in purple, bounded by a surface  $S$  shown as a blue wireframe mesh. Arrows pointing outwards from the surface represent the flux of a vector field through the boundary.

$$\int_W \text{div } \vec{F} dV = \int_S \vec{F} \cdot d\vec{A}$$

Pattern

$\int_{n\text{-dim'l object}} (\text{deriv. type object}) = \int_{\text{boundary of } n\text{-dim'l object}} ( )$

# Methods for Computing Line & Flux Integrals

## Line Integrals

① Parameterization

② FTCLI :

- Need  $\mathbf{v}\mathbf{f}$  to be \_\_\_\_\_
- Check using \_\_\_\_\_
- If domain has no holes &  $\text{curl} = 0$ ,  
find \_\_\_\_\_ & plug in  
endpoints of path.

③ Green's Thm

- $\mathbf{v}\mathbf{f}$  is \_\_\_\_\_ dimensional
- curve is \_\_\_\_\_
- curve & enclosed region in \_\_\_\_\_ of  $\mathbf{v}\mathbf{f}$
- compute \_\_\_\_\_ & integrate over region

④ Stokes' Thm

- $\mathbf{v}\mathbf{f}$  is \_\_\_\_\_ dimensional
- curve is \_\_\_\_\_
- curve & "spanning surface" in  
\_\_\_\_\_ of  $\mathbf{v}\mathbf{f}$
- compute (3D vector) curl & calculate  
flux through surface

# Flux & Flux Integrals

① Basic :  $\nabla f$  \_\_\_\_\_ & surface \_\_\_\_\_

- Dot product

② Compute flux integral directly

- $d\vec{A} = \hat{n} dA$ , ( $\vec{F}$  on  $S$ ), dot prod., ...

③ Divergence Thm

- surface must be \_\_\_\_\_
- surface & enclosed solid in \_\_\_\_\_
- calculate divergence & integrate over solid region enclosed by surface

④ Stokes' Thm

- $\nabla f$  must be a \_\_\_\_\_ field
- convert flux integral to must find
  - line int. along boundary  $\rightarrow$  \_\_\_\_\_
  - OR
  - flux integral through simpler surface