

Here is an example of a proof using strong induction.

Result. For every integer $n \geq 12$, there exist nonnegative integers a, b such that $n = 3a + 7b$.

Proof. We proceed by strong induction on $n \geq 12$.

Base Case: $n = 12$. In this case $n = 3(4) + 7(0)$. Since 4 and 0 are nonnegative integers, we see that n can be written in the desired form.

Inductive Step. Fix $n \geq 12$. Suppose that for all integers k in the range $12 \leq k \leq n$, there exist nonnegative integers a and b such that $k = 3a + 7b$. Now consider $n + 1$. (We aim to show that $n + 1$ is also of this form.)

Notice that if $n + 1 = 13$, then $n + 1 = 3(2) + 7(1)$, and if $n + 1 = 14$, then $n + 1 = 3(0) + 7(2)$. Thus in these two cases, $n + 1$ is in the desired form.

It remains to treat the case that $n + 1 \geq 15$ (in other words $n \geq 14$), so let $n + 1 \geq 15$. We see that

$$n + 1 = (n + 1) - 3 + 3 = (n - 2) + 3.$$

Since $n \geq 14$, $n - 2 \geq 12$. Thus, by the inductive hypothesis, there exist nonnegative integers a and b such that $n - 2 = 3a + 7b$. Now substituting for $n - 2$ in the expression for $n + 1$ gives us:

$$n + 1 = (3a + 7b) + 3 = 3(a + 1) + 7b.$$

Since $a + 1$ and b are nonnegative integers $n + 1$ is in the desired form. This concludes the proof of the inductive step.

By the strong principal of mathematical induction, we have shown that for all integers $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$. □

Note. The proof could be organized differently, treating the cases $n = 12$, $n = 13$, and $n = 14$ as a series of “base cases.” Then the inductive step would start by fixing $n \geq 14$, supposing that every integer k in the range $12 \leq k \leq n$ is of the form $3a + 7b$, where a and b are nonnegative integers, and aiming to show that $n + 1$ is also of this form.