For an estimator $\hat{\theta}$ we define the **mean squared error** to be:

$$\text{MSE}(\hat{\theta}) \;=\; E[(\hat{\theta} - \theta)^2]\,.$$

The mean squared error may be expressed in terms of the variance and bias of the estimator:

$$\text{MSE}(\hat{\theta}) \;=\; V(\hat{\theta}) \,+\, [\text{Bias}(\hat{\theta})]^2\,.$$

Recall that the bias of an estimator is $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$.

In this problem, we consider the case where the population distribution is normal, and the parameter of interest is the variance, σ^2. We consider all estimators of the form KS^2, where K is a constant and S^2 is the sample variance. The question is to determine which of these has the *smallest* mean squared error.

To minimize $\text{MSE}(KS^2)$, we use Calculus: calculate $\text{MSE}(KS^2)$, differentiate with respect to K, equate to zero, and solve for K. (To make sure this K corresponds to a minimum, not a maximum, of $\text{MSE}(KS^2)$, we can take another derivative and verify that it is positive.)

Along the way we will need to use the following facts:

1. For any random variable X, $V(X) = E(X^2) - E(X)^2$.
2. For any constant a and any random variable X, $E(aX) = aE(X)$.
3. As mentioned in the hint: $E((S^2)^2) = (n+1)\sigma^4/(n-1)$.
4. The sample variance S^2 is an unbiased estimator for the population variance σ^2, so $E(S^2) = \sigma^2$.