Unit 1: Ch. 1-5

Exercise 3.16 Modify to say: "Give a specific example of some group G, some elements $g, h \in G$, and some integer $n > 1$, such that $(gh)^n \neq g^n h^n$."

Exercise 3.25 Break this down into three cases: $n \geq 0$ (prove by induction), $n = -1$, and $n < 0$ (prove using the previous two cases.)

Exercise 3.48 Modify to say, "Let G be a group. Show that $Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$ is a subgroup of G. This subgroup is called the *center* of G."

Exercise 4.23 You may assume that G is finite, which implies that all elements of G are of finite order.

Exercise 4.27 It might help to prove Exercise 4.38 first.

Exercise 5.5 Instead of listing all the subgroups of S_4, list the *elements* of S_4. Then proceed to answer the rest of the questions.

Exercise 5.29 Instead of finding the center of D_8 and D_{10}, find the center of D_3 and D_4 and then *conjecture* the center of D_n.

Unit 2: Ch. 6, 9-12

Exercise 6.5 No computation is needed for (d) or (e).

Exercise 6.11 This is a great exercise, but, in the interest of time, just prove that conditions (a), (d), and (e) are equivalent.

Exercise 9.7 Use Exercise 6.14.

Exercise 9.18 You will want to prove the following lemma: "Let G, G', H, and H' be groups with $G \cong G'$ and $H \cong H'$. Then $G \times H \cong G' \times H'$."

Exercise 9.25 You will want to prove the following lemma: "Let G and H be abelian groups. Then $G \times H$ is abelian."

Exercise 9.27 You will want to prove the following lemma: "Let $\phi : G \to H$ be an isomorphism of groups. For any $a \in G$ and any $k \in \mathbb{Z}$, $\phi(a^k) = \big(\phi(a)\big)^k$." (To prove this, first prove the $k \geq 0$ case by induction, then prove the $k = -1$ case, and use the $k \geq 0$ case combined with the $k = -1$ case to prove the case $k < 0$.)

Exercise 11.11 To say that a homomorphism on a cyclic group is "completely determined" by its action on the generator of the group is to say that knowing the image of the generator is sufficient to determine the image of every other element of the group. One way to explicate this precisely is as follows: "Let $G = \langle a \rangle$ be a cyclic group with generator a, H be any group, and $\phi : G \to H$, $\psi : G \to H$ be group homomorphisms. If $\phi(a) = \psi(a)$, then $\phi = \psi$." You will want to prove the following lemma: "Let $\phi : G \to H$ be a homomorphism of groups. For any $a \in G$ and any $k \in \mathbb{Z}$, $\phi(a^k) = \big(\phi(a)\big)^k$." (To prove this, first prove the $k \geq 0$ case by induction, then prove the $k = -1$ case, and use the $k \geq 0$ case combined with the $k = -1$ case to prove the case $k < 0$.)

Exercise 13.6 You may want to invoke Exercise 4.39.

Exercise 13.18 Prove the following lemma, which is a strengthening of the result proven in Exercise 4.39: "Let G be a cyclic group of finite order n and m a positive divisor of n. Then G contains a unique subgroup of order m." To prove the lemma, let $G = \langle a \rangle$, and suppose a^k is of order m in G. Show that $\langle a^k \rangle = \langle a^d \rangle$ where $d = \gcd(n, k)$.

Unit 3: Ch. 14-17

Exercise 14.6 Hint: See Theorem 6.16 and Exercise 6.15: Two elements of S_n are conjugate if and only if they have the same cycle structure.

Exercise 15.2 Hint: See Theorem 6.16 and Exercise 6.15: Two elements of S_n are conjugate if and only if they have the same cycle structure.

Exercise 15.9 By "Show directly," we mean "Don't invoke Theorem 15.10."

Exercise 15.13 This should say, "Show that G must contain a *proper nontrivial* normal subgroup.'

Exercise 15.23 Don't merely invoke Lemma 15.6. Use this as an opportunity to prove the lemma in this special case.

Exercise 16.18 First prove the following lemma, "The identity in a ring with an identity is unique."

W 16.4 Consider the evaluation homomorphism $\phi_\alpha : C[a, b] \to \mathbb{R}$ given by $f \mapsto f(\alpha)$. Show that $\ker\phi_\alpha$ is a maximal ideal in $C[a, b]$.

D 17.3 Show that the factor ring $\mathbb{Z}_2[x]/\langle 1 + x^2 + x^3\rangle$ is a finite field with eight elements. Write out addition and multiplication tables for this field.

W 17.3 Show that the factor ring $\mathbb{Z}_3[x]/\langle 1 + x^2\rangle$ is a finite field with nine elements. Write out addition and multiplication tables for this field.

Unit 4: Ch. 21, 23

Exercise 21.1 Modify to say, "Show that each of the following numbers is algebraic over $\mathbb{Q}$." To prove that an element α in a field extension $E/\mathbb{Q}$ is algebraic, it suffices to find *any nonzero* polynomial in $\mathbb{Q}[x]$ having α as a root.

Exercise 23.6 Assume the polynomial is irreducible over a field of characteristic zero.

D 23.2 Show that $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{1+\sqrt{2}})/\mathbb{Q}(\sqrt{2})$ are normal but $\mathbb{Q}(\sqrt{1+\sqrt{2}})/\mathbb{Q}$ is *not* normal.

Exercise 23.21 The point of this exercise is to prove the *inclusion-reversing* property of the Galois correspondence described in the FTGT. For clarity, modify the exercise to say, "Let E/F be a finite, separable, normal extension of fields. Let $G = \mathrm{Gal}(E/F)$ be the Galois group of the extension. Let K, K' be intermediate fields between E and F, and let H, H' be corresponding subgroups of G, under the Galois correspondence. Show that $K \supset K'$ if and only if $H \subset H'$."

Exercise 13.12 Hint: Use the Correspondence Theorem and the Third Isomorphism Theorem.

Exercise 23.4 Hint: For some of these, it might be helpful to use what you know about cyclotomic poynomials. (Recall that cyclotomic polynomials are the irreducible factors of $x^n - 1$; they are also the minimal polynomials of roots of unity.)