Name: ______________________________

Read Section 23.2, focusing on the *examples*, particularly Examples 23.15 and 23.20 (which harken back to Examples 21.20, 23.4, 23.9), and the *statements* of results, rather than the proofs.

Reading Questions

1. **Fixed Fields.** Make sure you know the definition of the fixed field of a set of automorphisms of a given field.

2. **Normal Extensions.** The defining condition for a normal extension is not easily verified, and, in contrast to separable extensions, normal extensions are "special" rather than "typical." Splitting fields are normal.

   (a) A finite extension $F(\alpha)$ is normal if and only if the minimal polynomial of α over F has *all* its roots in $F(\alpha)$. For example,

      i. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is normal since both roots $\pm\sqrt{2}$ of x^2-2 are in $\mathbb{Q}(\sqrt{2})$, but

      ii. $\mathbb{Q}(\sqrt[3]{2})$ is *not* normal, since the complex roots $\sqrt[3]{2}\,\omega$ and $\sqrt[3]{2}\,\omega^2$ of x^3-2 are *not* in $\mathbb{Q}(\sqrt[3]{2})$.

   (b) A field extension E/F is finite, separable, and normal if and only if E is the splitting field of a separable polynomial over F.

3. **The Fundamental Theorem of Galois Theory.** Let E/F be a finite, separable, normal extension. (Note: if $F=\mathbb{Q}$ and E is the splitting field of an irreducible polynomial over $\mathbb{Q}$, then E/F is finite, separable, and normal.) Then

   (a) The intermediate fields K between E and F are in bijection with subgroups H of the Galois group $G=\mathrm{Gal}(E/F)$, via the correspondences:

      i. To every subgroup H of G, we associate an intermediate field, namely the fixed field of H, between E and F.

      ii. To every intermediate field K between E and F, we associate a subgroup, namely the Galois group of E/K, of G.

   (b) This bijective correspondence is *inclusion-reversing*:

      i. For subgroups H, H' of G, and corresponding fixed fields K, K', if $H \subset H'$, then $K \supset K'$.

      ii. For intermediate fields K, K' between E and F and corresponding Galois groups $H=\mathrm{Gal}(E/K)$ and $H'=\mathrm{Gal}(E/K')$, if $K \supset K'$, then $H \subset H'$.

   (c) For a subgroup H of G and corresponding intermediate field K between E and F,

   $$|H| = [E:K] \qquad \text{and} \qquad [K:F]=[G:H]$$

   (d) A subgroup H is normal in G if and only if the corresponding intermediate extension K/F is a normal extension. In this case,

   $$\mathrm{Gal}(K/F) \cong G/H$$

4. Is $\mathbb{Q}(\sqrt{5})$ normal over $\mathbb{Q}$? What about $\mathbb{Q}(\sqrt[3]{5})$? Explain.

5. Let $E = \mathbb{Q}(\sqrt{3}, \sqrt{5})$, $F = \mathbb{Q}$, and $G = \mathrm{Gal}(E/F) = \{\mathrm{id}, \sigma, \tau, \mu\}$, as in Examples 4, 9, 15, and 20.

   (a) Explicitly describe the correspondence between subgroups of G and intermediate fields between E and F, i.e. for each subgroup H of G, state the fixed field of H in E, and for each intermediate extension K between E and F, state the Galois group $\mathrm{Gal}(E/K)$.

   (b) Verify the *inclusion-reversing* property of the correspondence.

   (c) Verify that the orders and indices of the subgroups are equal to the corresponding degrees of intermediate extensions, as described in FTGT(c).

   (d) Which subgroups are normal? Which intermediate extensions are normal? Verify the isomorphism described in FTGT(d).

6. What struck you in this reading? What is still unclear? What remaining questions do you have?