We review complex powers (De Moivre's Theorem), complex roots, and roots of unity. See Section 3.3, especially pages 110-112, referencing the "Corrections and Modifications" document as needed.

1. Complex Powers

Recall that, when multiplying complex numbers, we **multiply moduli** and **add arguments**. Applying this principal to powers: for $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, $n \geq 1$, the modulus of z^n is the nth power of the modulus of z, and the argument of z^n is n times the argument of z. In exponential notation: $(re^{i\theta})^n = r^n e^{in\theta}$. This result is known as De Moivre's Theorem, and it can be proven by induction on n.

If z lies on the unit circle in the complex plane, the powers of z also do. If, in addition, the argument of z is $\theta = 2\pi/m$, for some positive integer m, then the powers of z repeat: $z^m = e^{m\cdot 2\pi i/m} = 1$ and $z^{m+k} = z^k$ for every positive integer k, so the powers of z are a finite set of evenly spaced points on the unit circle.

2. Complex Roots

To find the complex roots of a complex number, we again use polar form. It is important to recall that the argument is only defined up to adding multiples of 2π, i.e. $re^{i\theta} = re^{i(\theta+2\pi k)}$ for all $k \in \mathbb{Z}$.

Thus the nth complex roots of $re^{i\theta}$ are of the form $\sqrt[n]{r}\, e^{i(\theta+2\pi k)/n}$, for $0 \leq k < n$.

For example, the complex cube roots of 8 are $\alpha = 2$, $\beta = 2e^{2\pi i/3}$, and $\gamma = 2e^{4\pi i/3}$, as we can check:

- $\alpha^3 = 2^3 \;=\; 8$
- $\beta^3 = (2e^{2\pi i/3})^3 \;=\; 2^3\, e^{3\cdot 2\pi i/3} \;=\; 8e^{2\pi i} \;=\; 8$
- $\gamma^3 = (2e^{4\pi i/3})^3 \;=\; 2^3\, e^{3\cdot 4\pi i/3} \;=\; 8e^{4\pi i} \;=\; 8$

From the point of view of polynomials, α, β and γ are the three roots of $x^3 - 8$ in $\mathbb{C}$.

As a more interesting example, let's find the complex 5th roots of $z = 32i$. First we write z in exponential polar form: $z = 32e^{\pi i/2}$. Then we take the real fifth root of the modulus: $\sqrt[5]{32} = 2$ and divide the argument by five: $(\pi/2)/5 = \pi/10$. This gives us one complex 5th root of z: $\alpha = 2e^{\pi i/10}$. To get the arguments for the other four we simply add $2\pi k/5$ to the argument until we get redundancy:

$$\tfrac{\pi}{10}+\tfrac{2\pi}{5} = \tfrac{5\pi}{10} = \tfrac{\pi}{2}, \quad \tfrac{\pi}{2}+\tfrac{2\pi}{5} = \tfrac{9\pi}{10}, \quad \tfrac{9\pi}{10}+\tfrac{2\pi}{5} = \tfrac{13\pi}{10}, \quad \tfrac{13\pi}{10}+\tfrac{2\pi}{5} = \tfrac{17\pi}{10}, \quad \tfrac{17\pi}{10}+\tfrac{2\pi}{5} = \tfrac{21\pi}{10} = \tfrac{20\pi}{10}+\tfrac{\pi}{10} = 2\pi+\tfrac{\pi}{10}\ .$$

Thus the five complex 5th roots of $32i$ are:

$$\alpha = 2e^{\pi i/10},\ \ \beta = 2e^{5\pi i/10} = 2e^{\pi i/2},\ \ \gamma = 2e^{9\pi i/10},\ \ \delta = 2e^{13\pi i/10},\ \ \eta = 2e^{17\pi i/10}\ .$$

Notice that, from the first complex root, $\alpha = 2e^{\pi i/10}$, we may obtain all the others, by repeated multiplication by $\zeta_5 = e^{2\pi i/5}$, which is a complex number on the unit circle, one fifth of the way around the circle (counter-clockwise) from $1 + 0i$. We observe that $(\zeta_5)^5 = (e^{2\pi i/5})^5 = e^{2\pi i} = 1$,so ζ_5 is a fifth root of 1.

3. Roots of Unity

Given a positive integer n, we may seek to find the complex nth roots of 1. These are called the **nth roots of unity**. (Here we think of "unity" as a fancy way of saying "one.") Equivalently, the nth roots of unity are the complex roots of the polynomial $x^n - 1$.

From above, it is apparent that the complex nth roots of unity are $\zeta_n = e^{2\pi i/n}$ and its powers:

$$(\zeta_n)^k \;=\; e^{2\pi k i/n}\ ,$$

where k ranges over all positive integers, but due to redundancy, we often consider k in the range $0 \leq k < n$ (since $(\zeta_n)^n = 1 = (\zeta_n)^0$.)

Geometrically, the complex nth roots of unity lie on the unit circle in the complex plane, and are evenly spaced, starting from $1 + 0i$. For example, the complex 12th roots of unity are positioned on the unit circle like the numbers on a clock face.

An nth root of unity ζ is called a **primitive nth root of unity** if there is *no smaller positive* integer exponent such that ζ raised to that exponent is one, i.e. if m is a positive integer with $\zeta^m = 1$, then $m \geq n$.

For example, $\zeta = e^{2\pi i/3}$ is a sixth root of unity, since $\zeta^6 = e^{6 \cdot 2\pi i/3} = e^{4\pi i} = 1$, but ζ is *not* a primitive sixth root of unity, since $\zeta^3 = e^{3 \cdot 2\pi i/3} = e^{2\pi i} = 1$. In fact, ζ is a primitive cube root of unity, since there is no positive integer $d < 3$ such that $\zeta^d = 1$.

Note that $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity; geometrically, it is the first nth root of unity past $1 + 0i$ on the unit circle, proceeding counterclockwise, i.e. it is the nth root of unity with the smallest positive argument. Typically, it is not the only primitive nth root of unity, unless n is prime. For example, $\zeta_6 = e^{2\pi i/6}$ and $(\zeta_6)^5 = e^{10\pi i/6}$ are both primitive sixth roots of unity.

Corollaries 3.29 and 3.30 on pages 111-112 describe primitive roots of unity; note also the definition of the **Euler ϕ-function** at the bottom of page 111.

Theorem 3.32 (note the correction) states several useful properties of roots of unity, including the fact that the distinct nth roots of unity sum to zero. Make sure to take notes on this theorem. The examples (with illustrations!) before and after the theorem are also helpful.