Definition of Commutative Ring

A **commutative ring** is a set R with two binary operations, called **addition**,

$$+ \ : R \times R \to R \ , \quad \text{denoted by} \quad (a, b) \mapsto a + b \ ,$$

and **multiplication**,

$$\cdot \ : R \times R \to R \ , \quad \text{denoted by} \quad (a, b) \mapsto a \cdot b \ \text{ or } ab \ ,$$

satisfying the following eight properties:

(i) **Commutativity of Addition**:

For every $a, b \in R$, $a + b \ = \ b + a$.

(ii) **Existence of Additive Identity**:

There is an element of R, denoted 0 (or 0_R to distinguish from the integer 0), and called the **zero element** (or additive identity) of R, with the property: $a + 0 = a$, for any $a \in R$.

(iii) **Existence of Additive Inverses**:

For every $a \in R$, there is an element, denoted $-a$, and called the **negative** (or additive inverse) of a, such that $-a + a = 0$.

(iv) **Associativity of Addition**:

For all $a, b, c \in R$, $a + (b + c) \ = \ (a + b) + c$.

(v) **Commutativity of Multiplication**:

For all $a, b \in R$, $ab \ = \ ba$.

(vi) **Existence of Multiplicative Identity**:

There exists an element in R, denoted 1 (or 1_R to distinguish from the integer 1), and called the (multiplicative) **identity** of R, with the property that $1 \cdot a = a$ for all $a \in R$.

(vii) **Associativity of Multiplication**:

For all $a, b, c \in R$, $a(bc) \ = \ (ab)c$.

(viii) **Distributivity**:

For all $a, b, c \in R$, $a(b + c) \ = \ ab + ac$.