Please see http://classicalrealanalysis.info/com/TBB-Errata.php for a list of corrections to the textbook. Some are merely typos; others are more significant. What follows are my additional notes, modifications, exercises, and hints for you, organized by chapter.

Chapter 1

Field Axioms in 1.3 The field axioms **A.3** and **M.3** guarantee the existence of an additive identity, denoted 0, and a multiplicative identity, denoted 1. We should also explicitly say that $0 \neq 1$. This could be a tenth field axiom; alternatively we could include this condition in **M.3**, when defining the multiplicative identity, 1.

Exercise 1.3.0 Using only the field axioms, prove the following properties of the real numbers:

(a) For any $a \in \mathbb{R}$, $0 \cdot a = 0$. (Hint: Cite an axiom to justify the fact that $0 + 0 = 0$; then consider $(0 + 0) \cdot a$.)

(b) For any $a \in \mathbb{R}$, if $b \in \mathbb{R}$ satisfies $a + b = 0$, then $b = -a$. (Hint: Remember that $-a$ is defined to be a real number with the property that $a + (-a) = 0$. Note that this proves the uniqueness of additive inverses in $\mathbb{R}$.)

(c) For any $a, b \in \mathbb{R}$, $(-a)b = -(ab)$. (Hint: Show that $ab + (-a)b = 0$, and then use the uniqueness of additive inverses, proven in the previous part of this exercise.)

Exercise 1.4.1 Hint: As a warm-up, first focus on citing the order axioms appropriately and allowing yourself to use any algebraic properties of the real numbers that you know (including properties of subtraction). Then go back to your proof and fill in each step citing the appropriate field axiom or the property proven in Exercise 1.3.0.

Exercise 1.4.2 Hint: The key is to prove $0 < 1$. (You may want to use 1.3.0a at some point in this proof.) Then we consider $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, where 2 is defined as $1+1$, 3 is defined as $2+1$, 4 is defined as $3+1$, etc.. From the fact that $0 < 1$, prove that $2 > 1$, $3 > 1$, $4 > 1$, etc., and also $2 > 0$, $3 > 0$, $4 > 0$, etc.. (A formal proof would use induction, but for now, it's okay to be a bit informal in this regard.) Thus for all $n \in \mathbb{N}$, $n \geq 1$ and $n > 0$. From here it is just a hop, skip, and a jump to show that for all $n \in \mathbb{N}$, $n^2 \geq n$.

Exercise 1.7.3 Hint: Imitate the proof of the Archimedean Property. Consider the set $\{\frac{1}{n} | n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$. Clearly it is bounded below by zero, so it has a greatest lower bound, $\alpha \geq 0$. Assume (aiming for contradiction) that $\alpha > 0$. (To make the connection with the proof of the Archimedean Property, think of the x in that proof as $\frac{1}{\alpha}$. Then $x - 1$ is ....)

Theorem 1.15 It is a little hard to follow the last paragraph of this proof. Let me try to clarify it. So far in the proof we have real numbers x and y with $x < y$ and natural numbers n and m with

$$ny > nx + 1 \quad \text{and} \quad m \leq nx + 1 < m + 1 .$$

From the fact that $nx + 1 < m + 1$, we can see $nx < m$ and $x < m/n$, since $n > 0$. From the facts that $m \leq nx + 1$ and $nx + 1 < ny$, we see that $m < ny$, i.e. $m/n < y$. Thus we have found a rational number m/n in the interval (x, y); since the interval (x, y) was arbitrary, this proves that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Chapter 2

Example 2.10 We fill in some details to clarify this example. Recall that we wish to find a bound N that will ensure that as long as $n \geq N$,

$$\frac{n^2+1}{n+1} \geq M. \tag{$*$}$$

It is important to note that we do not need the smallest such N. First we rewrite the left side using long division of polynomials. Looking at the leading term of the numerator and denominator, we see that

$$\frac{n^2+1}{n+1} = n + \frac{(\text{something})}{n+1}.$$

The "something" will be $(n^2+1) - n(n+1) = -n+1$. Therefore,

$$\frac{n^2+1}{n+1} = n + \frac{(-n+1)}{n+1} = n - \frac{n}{n+1} + \frac{1}{n+1}.$$

Thus the inequality $(*)$ is equivalent to

$$n - \frac{n}{n+1} + \frac{1}{n+1} \geq M. \tag{$**$}$$

First notice that since $1/(n+1) > 0$,

$$n - \frac{n}{n+1} \geq M \quad \Rightarrow \quad n - \frac{n}{n+1} + \frac{1}{n+1} \geq M.$$

Moreover, since $n/(n+1) < 1$,

$$n - 1 \geq M \quad \Rightarrow \quad n - \frac{n}{n+1} \geq M.$$

The former inequality is clearly equivalent to $n \geq M+1$. So we choose $N = M+1$, since if $n \geq M+1$, then the inequality $(**)$, which is equivalent to the inequality $(*)$, is guaranteed to be true.

Exercise 2.9.6 The symbol "$\supset$" is used to denote set containment: for two sets A and B, $A \supset B$ means every element of B is an element of A. In this case, we may say that A is a *superset* of B; equivalently, B is a subset of A, which is denoted $B \subset A$. See Appendix A.2.1, Set Notation. (Note that some authors use these symbols differently; they use the symbol "$\subset$" to indicate a *proper* subset (a subset which is not equal to the whole set) and, similarly, the symbol "$\supset$" to indicate proper containment.)

Convergence of Subsequence. Let $\{s_n\}$ be a sequence. To say that a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ converges to a number L means:

> For all $\varepsilon > 0$, there is $K \in \mathbb{N}$ such that $|s_{n_k} - L| < \varepsilon$ for all $k \geq K$.

Moreover, we claim that the existence of a such a convergent subsequence $\{s_{n_k}\}$ implies:

> For all $\varepsilon > 0$, and for all $m \in \mathbb{N}$, there is $n \geq m$ such that $|s_n - L| < \varepsilon$.

To see why this is so, take any $\varepsilon > 0$. Since $\{s_{n_k}\}$ converges to L, there is a $K_\varepsilon \in \mathbb{N}$ such that $|s_{n_k} - L| < \varepsilon$ for all $k \geq K_\varepsilon$. Let $m \in \mathbb{N}$. Since $\{n_k\}$ is an increasing sequence of natural numbers, it is not bounded above and, in fact, there is a $K_m \in \mathbb{N}$ such that $n_k \geq m$ for all $k \geq K_m$. Let $k = \max\{K_\varepsilon, K_m\}$, and let $n = n_k$. Then $n \geq m$ and $|s_n - L| < \varepsilon$. Thus we have shown that, for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$, there is a natural number $n \geq m$ such that $|s_n - L| < \varepsilon$.

Chapter 4

Exercise 4.3.15 Modify this exercise to say, "Let E be a nonempty, bounded set of real numers. Show that if E is closed, $\min(E)$ and $\max(E)$ both exist, but if E is open, neither $\min(E)$ nor $\max(E)$ exist (although $\sup(E)$ and $\inf(E)$ do.)"

Theorem 4.33 We provide an alternate proof, which does not rely on Section 4.5.3.

Theorem (Heine-Borel)**.** A set A in $\mathbb{R}$ is closed and bounded if and only if it has the Heine-Borel Property, namely: every open cover of A can be reduced to a finite subcover.

Proof. We begin with the "forwards direction," showing that a closed and bounded subset of $\mathbb{R}$ has the Heine-Borel Property. Let A be a closed and bounded set of real numbers, and let $\mathcal{U}$ be an open cover of A.

First we treat the case that A is a closed and bounded interval, $[a, b]$. Consider the set E consisting of all $x \leq b$ such that $[a, x]$ is covered by a finite subcover of $\mathcal{U}$.

Certainly $a \in E$, since it can be covered by a single set in $\mathcal{U}$; therefore E is nonempty. Moreover, E is bounded above by b, so E has a supremum; let $c = \sup E$. (Note that c must be in $[a, b]$: $c \geq a$ since $a \in E$ and c is an upper bound for E; $c \leq b$ since b is an upper bound for E and c is the *least* upper bound for E.)

Since $c \in [a, b]$, there is an open set $U_0 \in \mathcal{U}$ such that $c \in U_0$. Since U_0 is open, there is an $\varepsilon > 0$ such that the open interval $(c - \varepsilon,\ c + \varepsilon)$ is contained in U_0.

Now since $c - \varepsilon$ is less than c, and c is the least upper bound for E, this means that $c - \varepsilon$ is *not* an upper bound for E; in other words, there is some $x \in E$ such that $x > c - \varepsilon$.

Since $x \in E$, there is a finite collection $\{U_1, U_2, \ldots, U_n\} \subset \mathcal{U}$ covering $[a, x]$, by the definition of E. Consequently the finite collection $\{U_0, U_1, U_2, \ldots, U_n\}$ covers $[a, c + \varepsilon)$. In particular, since this collection covers $[a, c]$ and since $c \leq b$, we must have $c \in E$.

We claim that in fact $c = b$. If, to the contrary, $c < b$, let $\delta = \min\{b - c,\ \varepsilon/2\}$, which is positive. Then $[a, c + \delta]$ is covered by the finite collection $\{U_0, U_1, U_2, \ldots, U_n\}$ and $c + \delta \leq b$, so $c + \delta \in E$. But since $\delta > 0$, this contradicts that c is the least upper bound for E. Therefore $c = b$, implying $b \in E$, which means that $[a, b]$ has the Heine-Borel property.

Now we consider the more general case, where A is a closed and bounded set in $\mathbb{R}$ but not necessarily an interval. In this case, A is contained in some closed, bounded interval, $[a, b]$.

With $\mathcal{U}$ as above, namely an open cover of A, let $\mathcal{U}^* = \mathcal{U} \cup \{\bar{A}\}$, where $\bar{A}$ denotes the complement of A, which is open. Since every element of $\mathbb{R}$ is either in A or not in A, the collection $\mathcal{U}^*$ is an open cover of $\mathbb{R}$ and thus of $[a, b]$.

By the previous case, there is a finite subset of $\mathcal{U}^*$ which covers $[a, b]$ and thus A. If this finite subset does not include $\bar{A}$, then it is a finite subset of $\mathcal{U}$ that covers A, and we are done. On the other hand, if the finite subset *does* include $\bar{A}$, then removing $\bar{A}$ from the subset yields a finite subset of $\mathcal{U}$ that covers A, since no point in A is in $\bar{A}$. Thus we have shown that $\mathcal{U}$ reduces to a finite subcover of A, so A has the Heine-Borel Property. This completes the "forwards direction" of the proof.

To show the "backwards direction," we must show that a set A of real numbers having the Heine-Borel Property must be closed and bounded. Let A be a set of real numbers having the Heine-Borel Property.

To show that A is closed, it suffices to show that its complement, $\bar{A}$, is open. We will show that every point in $\bar{A}$ is an interior point of $\bar{A}$. Take any $x \in \bar{A}$.

We construct an open cover for A by placing a small interval around each point of A, as follows. Take $y \in A$, let $\varepsilon_y = \frac{1}{3}|x-y|$, and let $U_y = (y-\varepsilon_y,\, y+\varepsilon_y)$. (Note that this means that U_y and $(x-\varepsilon_y,\, x+\varepsilon_y)$ are disjoint.)

Then $\mathcal{U} = \{U_y \,|\, y \in A\}$ is an open cover of A. Since A has the Heine-Borel Property, $\mathcal{U}$ reduces to a finite subcover, i.e. there is a finite set of points $y_1, \ldots, y_n \in A$ such that $\{U_{y_1}, \ldots, U_{y_n}\}$ covers A.

Let $\varepsilon = \min\{\varepsilon_{y_i} \,|\, 1 \leq i \leq n\}$. This ensures that $(x-\varepsilon,\, x+\varepsilon)$ is small enough that it does not intersect any of the U_{y_i}, in the finite subcover, i.e. $(x-\varepsilon,\, x+\varepsilon) \cap U_{y_i} = \emptyset$ for all $1 \leq i \leq n$. In particular this means that $(x-\varepsilon,\, x+\varepsilon) \cap A = \emptyset$, i.e. $(x-\varepsilon,\, x+\varepsilon) \subset \bar{A}$ and x is an interior point of $\bar{A}$. Since x was an arbitrary point in $\bar{A}$, we can conclude $\bar{A}$ is open and A is closed.

Finally, we must show that A is bounded. We construct an open cover of A consisting of intervals of the form $(x-1, x+1)$ for every $x \in A$. Since A has the Heine-Borel Property, this open cover reduces to a finite subcover, i.e. there is a finite set of points $x_1, \ldots, x_n$ such that every point in A is within a distance of one from one of these points: for every $x \in A$, there is some $1 \leq i \leq n$ such that $|x - x_i| < 1$. Let $m = \min\{x_i - 1 \,|\, 1 \leq i \leq n\}$ and $M = \max\{x_i + 1 \,|\, 1 \leq i \leq n\}$. Then every x in A satisfies $m < x < M$, i.e. A is bounded.

We have now shown that any set A in $\mathbb{R}$ with the Heine-Borel Property is closed and bounded, completing proof of the theorem. $\square$

Exercise 4.5.14 Given two sets of real numbers E and K, first show directly that if E is closed and K is closed and bounded, the intersection $E \cap K$ is closed and bounded. Then show directly that if E is closed and K has the Bolzano-Weierstrass Property, then $E \cap K$ has the Bolzano-Weierstrass Property. (Do *not* use Theorem 4.21, which says that a set of real numbers is closed and bounded if and only if it has the Bolzano-Weierstrass Property.)

Chapter 5

Exercise 5.4.11 In this problem, assume that f is a function defined on a neighborhood of x_0 (i.e. $f : E \to \mathbb{R}$ is a function and $x_0 \in \operatorname{int}(E)$.)

Exercise 5.7.2 An alternate way to prove this is to use the Heine-Borel Property and Theorem 5.37. Such a proof generalizes beyond functions on the real line.

Chapter 7

Exercise 7.2.17 Modify to say: "Suppose a function has both a right-hand and a left-hand derivative at a point. Show that, if both one-sided derivatives are finite at that point, then f is continuous at the point. Then give an example to show that if one of the one-sided derivatives is infinite at that point, then f may not be continuous at the point."

Exercise 7.6.2 Modify to say: "Use Rolle's Theorem to explain why the cubic equation $x^3 + \alpha x^2 + \beta = 0$ cannot have more than one *positive* solution when $\alpha > 0$."

Exercise 7.7.4 For (d), you may assume that trig functions are continuous where defined.

Chapter 13

Exercise 13.1.2 For (a), you must show the following:

- The set $\mathcal{C}[0,1]$ together with the operation of addition of functions (namely: for functions f, g on a common domain, their sum is defined by: $(f+g)(x) = f(x) + g(x)$ for all x in the domain) has the following properties:
  1. (Closure.) For all $f, g \in \mathcal{C}[0,1]$, $f + g \in \mathcal{C}[0,1]$.
  2. (Associativity.) For all $f, g, h \in \mathcal{C}[0,1]$, $(f+g)+h = f+(g+h)$.
  3. (Identity.) There exists a function $I \in \mathcal{C}[0,1]$ such that $I + f = f + I = f$ for all $f \in \mathcal{C}[0,1]$.
  4. (Inverses.) For all $f \in \mathcal{C}[0,1]$, there is a function $g \in \mathcal{C}[0,1]$ such that $f + g = g + f = I$.
  5. (Commutativity.) For all $f, g \in \mathcal{C}[0,1]$, $f + g = g + f$.
- The set $\mathcal{C}[0,1]$ together with the operation of multiplication by constants (namely: for a function f and a constant c, the function cf is defined by $(cf)(x) = c \cdot f(x)$ for all x) has the following properties:
  6. (Closure.) For all $c \in \mathbb{R}$ and $f \in \mathcal{C}[0,1]$, $cf \in \mathcal{C}[0,1]$.
  7. (Identity.) For all $f \in \mathcal{C}[0,1]$, $1f = f$.
  8. (Associativity.) For all $c_1, c_2 \in \mathbb{R}$, $c_1(c_2 f) = (c_1 c_2) f$.
  9. (Distributivity over Vector Sums.) For all $c \in \mathbb{R}$ and $f, g \in \mathcal{C}[0,1]$, $c(f+g) = cf + cg$.
  10. (Distributivity over Scalar Sums.) For all $c_1, c_2 \in \mathbb{R}$ and $f \in \mathcal{C}[0,1]$, $(c_1 + c_2) f = c_1 f + c_2 f$.

In parts (b),(c), and (d), you must prove properties of $\mathcal{C}[0,1]$ involving function multiplication (namely: for functions f and g on a common domain, fg is given by: $(fg)(x) = f(x) \cdot g(x)$ for all x in the domain).