Name: ______________________________

Read Section 1.1, Domains in the Plane.

This section provides an introduction to the book (Paragraph 1.1.0), reviews the basic algebraic and geometric structure of the plane (1.1.1 and 1.1.2) and introduces some topological concepts that we will need (1.1.3 and 1.1.4). Read carefully, marking up your copy of the text and taking notes.

Reading Questions

1. Reread Paragraph 1.1.1, and brush up on your vector arithmetic by trying Exercise 1.

2. Reread Paragraph 1.1.2, and brush up on your plane geometry by trying Exercises 1, 2, and 3.

3. Reread Paragraph 1.1.3. Give examples (pictures suffice) of the following types of sets in the plane:
   (a) an open set

   (b) a closed set

   (c) a connected set

   (d) a disconnected set

4. Make sure you have the definition of a **domain** word for word in your notes. Give an example of one set in the plane that is a domain and one that is not.

5. Reread Paragraph 1.1.4. What is the boundary of the right half-plane ($x > 0$)? Is the right half-plane a bounded set?

6. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Section 1.2, Plane Curves. This section reviews parametric curves in the plane (1.2.0-1.2.3) and the dot product of vectors (1.2.5) and introduces Jordan curves, Jordan domains, and the outward normal vector for a curve (1.2.4, 1.2.6).

Reading Questions

1. Reread 1.2.1 and 1.2.2 and try Exercise 1 for Paragraph 1.2.2.

2. Draw one simple closed curve and one curve that is neither simple, nor closed.

3. When a curve Γ has been parametrized by arc length (as described in 1.2.3), what is the length of the tangent (velocity) vector?

4. Make sure you have the definitions for **Jordan curve** and **Jordan domain** (from 1.2.4) in your notes, word for word.

5. After rereading 1.2.5, brush up on the dot product by trying Exercise 1.

6. Paragraph 1.2.6 introduces the outward normal vector $N(s)$ for a Jordan domain Ω at a point $\sigma(s)$ on the boundary of Ω, where σ is an arclength parametrization of the boundary. What three features are used to define $N(s)$?

7. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Section 1.3, Differential Calculus in Two Variables. This section reviews some basic ideas from multivariable calculus: continuity, differentiability and linear approximation, directional derivatives, and the gradient (1.3.0-1.3.3) and introduces the outward normal derivative and the Laplacian.

Reading Questions

1. Suppose u is a continuously differentiable function on the plane, z is a point in the plane, and V is a vector in the plane.

   (a) Is the directional derivative $u'(z;V)$ a scalar or a vector?

   (b) Is the gradient $\nabla u(z)$ a scalar or a vector?

   (c) How can you compute the directional derivative $u'(z,V)$ from the gradient ∇u?

   (d) The outward normal derivative $\partial u/\partial n$ at a point on the boundary of a Jordan domain is the directional derivative of u in the direction of what vector?

2. For the function $u(x,y) = xe^y$, and $z_0 = (1,0)$ and $z_1 = (0,1)$ on the boundary of the unit disc $\Omega = D(0;1)$, compute:

   (a) ∇u

   (b) $\nabla u(z_0)$

   (c) $\nabla u(z_1)$

(d) $(\partial u/\partial n)(z_0)$

(e) $(\partial u/\partial n)(z_1)$

(f) Δu

(g) Is u harmonic?

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 1.4, Integral Calculus in the Plane. This section reviews line integrals, double integrals, and Green's Theorem (1.4.0-1.4.3) and introduces Green's Identities.

Reading Questions

1. Reread Paragraph 1.4.1. Theorem 8 is also known as the Fundamental Theorem of Calculus for Line Integrals.

   (a) Consider $u(x, y) = 2xy$. Compute the exact differential $du = u_x\, dx + u_y\, dy$.

   (b) Let Γ be the line segment from $z_0 = (0, 0)$ to $z_1 = (1, 2)$. Parametrize Γ using a function $\alpha(t) = \langle x(t), y(t)\rangle$ such that $\alpha(0) = z_0$ and $\alpha(1) = z_1$.

   (c) Using your parametrization, compute $dx = x'(t)dt$ and $dy = y'(t)dt$.

   (d) Compute the line integral $\int_\Gamma du$ directly, using your parametrization.

   (e) Compute the line integral $\int_\Gamma du$ using the Fundamental Theorem of Line Integrals.

2. Let Ω be the unit square with corners $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, traversed in that order. Use Green's Theorem to compute the line integral:

$$\int_{\partial\Omega} \left[\, 3xy\,dx \;+\; x^2\,dy \,\right]$$

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 2.1, Basic Properties of Harmonic Functions.

Reading Questions

1. Reread the examples of harmonic functions.

   (a) Check the claims in Example 3 for yourself, i.e. show that $x^2 - y^2$ and xy are harmonic but $x^2 + y^2$ is not.

   (b) Check the claims in Example 5 for yourself, i.e. show that the sum of two harmonic functions is harmonic, the scalar multiple of a harmonic function is harmonic, but the product of two harmonic functions is not necessarily harmonic.

2. Try Exercise 1(a)-(d). You may reference the examples given in the section or check harmonicity directly by applying the Laplacian.

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read 2.2, Harmonic Functions as Steady-State Temperatures.

Reading Questions

1. Reread Paragraphs 2.2.1 and 2.2.2. Suppose u is a function in $C^2(\Omega)$.

   (a) By Theorem 1(i), if u is harmonic on Ω, then $\int_{\partial\Omega}(\partial u/\partial n)\,ds$ = ______ , i.e. the net flux of u across the boundary of Ω is ______________ .

   (b) As argued in 2.2.2, from physical reasons, if u represents a steady state temperature in $\overline{\Omega}$, then, for any simple closed curve Γ that lies in Ω and whose interior also lies in Ω, the net heat flow across Γ must be ______________ , i.e. $\int_{\Gamma}(\partial u/\partial n)\,ds$ = ______ .

   (c) State Theorem 2.

   (d) What does Theorem 2 add to Theorem 1? (There are two things.)

   (e) Putting the argument in 2.2.2 together with Theorem 2, we can conclude that if u is a steady state temperature, then u is ________________ .

2. Reread 2.2.3. Briefly state the three conjectures about harmonic functions, and explain why they are plausible for physical reasons.

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Math 419, 2.3-2.4 Mean Value Properties and the Maximum Principle

Name: ______________________________

Read 2.3, Mean-Value Properties of Harmonic Functions and 2.4, The Maximum Principle.

Reading Questions

1. Reread Section 2.3. Suppose u is a harmonic function on a domain Ω, ζ is a point in Ω, D is a disc around ζ such that D and its boundary $\partial D = C$ are both inside Ω.

   (a) Draw a picture illustrating this set-up. Your picture should include Ω, ζ, D and C.

   (b) By the Circumferential Mean Value Theorem, the value of u at ζ is equal to the average value of u along what curve?

   (c) By the Solid Mean Value Theorem, the value of u at ζ is equal to the average value of u over what region?

2. Theorem 3 provides a sort of converse to the Circumferential Mean Value Theorem: every function satisfying the ______________________________ property is ______________ .

3. Reread Section 2.4. Suppose u is a nonconstant harmonic function on a domain Ω.

   (a) What does the Strong Maximum Principle say about u?

   (b) Assuming moreover that Ω is bounded and that u is continuous on $\partial\Omega$, what does the Weak Maximum Principle say about u?

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read 2.5, Harnack's Inequality and Liouville's Theorem and Appendix 2.1, On Differentiation under the Integral Sign and an Application: Harmonic Functions are C^∞.

Reading Questions

1. Reread 2.5.1.

   (a) What theorem is invoked twice in the proof of Harnack's Inequality?

   (b) Harnack's inequality gives an upper bound for the values of a nonnegative harmonic function on a disc. It is "noteworthy" that the upper bound only depends on the value of u itself at the center of the disc, *not* also on what? (See the comment after the proof.)

2. Suppose u is a nonconstant harmonic function on $\mathbb{R}^2$. What does Liouville's Theorem say about how large or how small the values of u can be?

3. What is the application given in Appendix 2.1? Why is it remarkable?

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read 3.1, Complex Numbers.

Reading Questions

1. Reread Paragraph 3.1.1, and practice addition, subtraction, and multiplication of complex numbers by trying Exercise 1 at the end of the paragraph.

2. Reread Paragraph 3.1.2, and practice thinking geometrically about complex numbers by trying Exercise 1 at the end of the paragraph.

3. Reread Paragraph 3.1.3, and practice inversion and division of complex numbers by trying Exercise 1 at the end of the paragraph.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____________________________________

Read 3.2, Complex Analytic Functions.

Reading Questions

1. Reread the part of Paragraph 3.2.1 entitled "Pictures."

   (a) Why do we not graph complex-valued functions of a complex variable?

   (b) Consider the function $f(z) = 2z$, and let $\Omega = D(0, 1)$. Sketch the set Ω in the z-plane and $f(\Omega)$ in the w-plane.

2. Reread the part of Paragraph 3.2.1 entitled "Real and Imaginary Parts." Check that for the function $f(z) = z^2$, we have $\mathrm{Re}(f) = x^2 - y^2$ and $\mathrm{Im}(f) = 2xy$, where, as usual, $z = x + iy$.

3. Reread Paragraph 3.2.2. Find the complex derivatives of the following functions:

   (a) $f(z) = (2+i)z$

   (b) $f(z) = (8-i)(2z-1)^7$

   (c) $f(z) \;=\; \dfrac{z}{i+z^2}$

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read 3.3, The Cauchy-Riemann Equations.

Reading Questions

1. Fill in the blanks:
   (a) If f is a complex analytic function on a domain Ω and u and v are its real and imaginary parts respectively, then u_x = __________ and u_y = __________ .
   (b) If u and v are in $C^1(\Omega)$ and satisfy the differential equations above for all z in a domain Ω, then the complex function $f = u + iv$ is ____________________ in Ω.

2. Consider the complex function $f(z) = i + 2z$.
   (a) Find the real and imaginary parts u and v of f.
   (b) Show that u and v satisfy the Cauchy-Riemann equations.

3. What are the Cauchy-Riemann equations in polar coordinates?

4. Show that $f(z) = |z|$ is *not* analytic by converting it to polar coordinates and using the Cauchy-Riemann equations in polar coordinates.

5. What struck you in this reading? What is still unclear? What remaining questions do you have?

Math 419, 3.4-I, The Exponential Function

Name: ______________________________

Read Paragraph 3.4.0, Introduction to the Exponential and Related Functions, and Paragraph 3.4.1, The Exponential Function.

Reading Questions

1. (a) How do we define the complex exponential function $\exp(z)$ for $z = x + iy$?

   (b) What does it mean to say that $\exp(z)$ is *entire analytic*?

   (c) What is the period of $\exp(z)$?

2. Practice computing with the complex exponential by trying Exercise 1 at the end of Paragraph 3.4.1.

3. Reread the part about the geometric meaning of a product of complex numbers (second half of page 123 and the figure on page 124). Fill in the blanks: For two complex numbers z and w,

   (a) the modulus, $|zw|$, of their product is equal to ...

   (b) and the argument, $\arg(zw)$, of their product is equal to ...

4. Practice using polar form by trying Exercise 7 at the end of Paragraph 3.4.1.

5. (a) Given a nonzero complex number z, how many n^{th} roots does it have?

   (b) What term do we use for the n^{th} roots of 1?

   (c) List the fourth roots of unity. (See Figure 3.6.)

6. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Paragraph 3.4.2, The Logarithm.

> ****Note.** There is a typo in Exercise 6 at the end of Paragraph 3.4.2. It should say $f(z) = e^{\pi z}$.

Reading Questions

1. Why is it impossible to define an inverse function for the complex exponential function?

2. Reread the discussion of the mapping properties of the exponential function (page 128). As in the text, let S be the horizontal strip $S = \{(x, y) \mid -\pi < y \leq \pi\}$.

   (a) To what region in the w-plane does the complex exponential map the *left half* $(x < 0)$ of the horizontal strip S? (Describe in word and draw a picture.)

   (b) To what region in the w-plane does the complex exponential map the *right half* $(x > 0)$ of the horizontal strip S? (Describe in word and draw a picture.)

   (c) To what curve in the w-plane does the complex exponential map the line segment $(x = 0)$ in the middle of the horizontal strip S? (Describe in word and draw a picture.)

3. Let $z = re^{i\theta}$, where $r > 0$ and $-\pi < \theta < \pi$. What is $\log(z)$?

4. Practice computing the complex logarithm by trying Exercise 1 at the end of Paragraph 3.4.2.

5. What struck you in this reading? What is still unclear? What remaining questions do you have?

Math 419, 3.4-III, Complex Trigonometric Functions and Complex Exponents

Name: ____

Read Paragraph 3.4.3, Complex Trigonometric Functions, and Paragraph 3.4.4, Complex Exponents.

Reading Questions

1. (a) What are the definitions of the complex sine and cosine?

   (b) Compute $\cos(i\pi)$.

2. For complex numbers z and c with $z \neq 0$, how do we define z^c?

3. Practice computing with complex exponents by trying Exercise 1 at the end of Paragraph 3.4.4.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 3.5, The Harmonic Conjugate.

Reading Questions

1. Reread the first paragraph of Section 3.5.

   (a) What are the two methods we have used so far to show that a given function is analytic?

   (b) What is the fourth method (to be discussed in the future)?

2. In this section we see how to construct analytic functions from harmonic functions. What needs to be true *about the domain* on which a function u is harmonic in order for us to be able to construct a harmonic conjugate v (so that we have an analytic function $f = u + iv$.)

3. Practice constructing harmonic conjugates by trying Exercise 1 at the end of Section 3.5.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Appendix 3.1, The Riemann Surface for $\log z$.

> ****Note.** My book arrived without pages 144 and 145. (I guess you get what you pay for.) If your book is missing these pages too, I might recommend using a free online book preview service like Google books to read these pages.

Reading Questions

1. Recall that, to resolve the ambiguity in the complex logarithm, we removed the nonpositive x-axis from the complex plane, and defined branches of the logarithm on the resulting slit plane. Riemann's approach to the complex logarithm is the "opposite" of this approach in what way?

2. After defining the Riemann surface X, we define the logarithm on X. The text lists four important properties of this function (in addition to the fact that it can be considered as an inverse to a version of the complex exponential map) in the first two paragraphs of page 144. What are these four properties?

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Discussion Questions

1. Check that $\log : X \to \mathbb{C}$ given on page 143 is (a) single-valued, (b) one-to-one, and (c) onto $\mathbb{C}$.
2. Define a suitable map $\exp : \mathbb{C} \to X$ to obtain an inverse function for $\log : X \to \mathbb{C}$.
3. Construct a natural Riemann surface for $z^{1/2}$.

Name: ____

Read Section 4.1, The Complex Line Integral.

Reading Questions

1. Reread Paragraph 4.1.1, and imitate the example to compute $\int_\Gamma z\,dz$, where Γ is the line segment from the origin to the point $1+i$.

2. Reread Paragraph 4.1.2. Use Theorem 1 to evaluate the integral in the previous problem. (You can imitate the second approach in Example 1.)

3. Reread Paragraph 4.1.3. We will apply the ML-Inequality to $\int_\Gamma z\,dz$ where Γ is as above.
   (a) What is the maximum value of $|z|$ on Γ? Call this maximum value M.

   (b) What is the length of the curve Γ? Call this length L.

   (c) Find an upper bound for $\left|\int_\Gamma z\,dz\right|$ using the ML-inequality.

   (d) Does this upper bound agree with your answers from the previous two problems?

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Section 4.2 The Cauchy Integral Theorem.

Reading Questions

1. This section contains four versions of the Cauchy Integral Theorem. Each has essentially the same conclusion, but the hypotheses differ. State each version here.

2. Explain why the Cauchy Integral Theorem *cannot* be used to evaluate $\int_{C(0,1)} \frac{1}{z}\, dz$.

3. Corollary 3 is an important consequence of the Cauchy Integral Theorem and is often used in "contour-shifting" arguments. State Corollary 3 here.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Section 4.2 The Cauchy Integral Formula.

****Note**. There is a typo in Exercise 4 at the end of this section. The integral should be $2\pi i(e-1)$.

Reading Questions

1. State the Cauchy Integral Formula. (Make sure to include all the hypotheses.)

2. Use the Cauchy Integral Formula to evaluate the following integrals. In each case Γ is the circle of radius 4 centered at the origin.

   (a) $\displaystyle\int_\Gamma \frac{e^z}{z-1}\,dz$

   (b) $\displaystyle\int_\Gamma \frac{e^z}{z-i\pi}\,dz$

   (c) $\displaystyle\int_\Gamma \frac{\sin(z)}{z}\,dz$

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Math 419, 4.4, 4.10 Higher Derivatives of Analytic Functions and Morera's Theorem

Name: ______________________________

Read Section 4.4 Higher Derivatives of Analytic Functions and Section 4.10, Morera's Theorem.

Reading Questions

1. State Theorem 7 in Section 4.4.

2. Give an example of a real-valued function f such that f' exists and is continuous on $\mathbb{R}$, but f' is not differentiable on all of $\mathbb{R}$.

3. State Morera's Theorem.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Sections 4.5-4.8 which discuss several corollaries of the harmonicity of the real and imaginary parts of an analytic function and related results. (You may omit the appendix to 4.5, on the Dirichlet problem.)

Reading Questions

1. Why were we not able to prove the harmonicity of the real and imaginary parts of an analytic function immediately after the Cauchy-Riemann Equations in Section 3.3? (You may find it helpful to reread pages 118 and 119.)

2. In Section 4.6, we prove the Circumferential and Solid Mean Value Theorems for analytic functions. These theorems can be proven just using the harmonicity of the real and imaginary parts of analytic functions, which means that they are applicable to a broader class of functions than analytic functions. (See the comment after the proof of the CMVT.) Can you think of an example of a complex-valued function that is *not* analytic but for which these theorems would still apply?

3. In Section 4.7, we prove the Maximum Modulus Principle. Why doesn't it make sense to try to prove a "Maximum Principle" (without the modulus) for complex-valued functions? (See the first paragraph of the section.)

4. In Section 4.8, we prove the Fundamental Theorem of Algebra (FTA) using the Maximum Modulus Principle. Comment 2 (at the bottom of page 186 and top of 187) explains what the FTA does and does not do for us. Explain this in your own words.

5. What struck you in this reading? What is still unclear? What remaining questions do you have?

Math 419, 4.9, 4.11 Liouville's Theorem and The Cauchy Inequalities

Name: ______________________________

Read Section 4.9, Liouville's Theorem, and Section 4.11, The Cauchy Inequalities for $f^{(n)}(z_0)$.

Reading Questions

1. State Liouville's Theorem for complex analytic functions.

2. In these sections we see two proofs of Liouville's Theorem for complex analytic functions.

   (a) In Section 4.9, the proof given uses the harmonic theory and thus applies to a broader class of functions than complex analytic functions. What is this broader class of functions? The proofs of what other theorems in this chapter would also apply to this class of functions?

   (b) In Section 4.11, the proof for Liouville's Theorem uses the Cauchy Inequalities. Two specific results from this chapter are invoked by name in the proof of the Cauchy Inequalities. What are they?

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____________________

This is a long reading. **Skim through** Section 5.1, Sequences and Series, and Section 5.2, Power Series. **Focus on Paragraphs 5.2.0 and 5.2.1**, which discuss a simple example (the complex geometric series) and the idea of the disc of convergence in general.

Reading Questions

1. Reread Paragraph 5.2.0.

   (a) For what complex numbers z does the geometric series $1 + z + z^2 + z^3 + \dots$ converge?

   (b) When the geometric series does converge, what the closed form for its sum?

   (c) Use the closed form for the sum of the geometric series to calculate the following. Write your answers in standard form $a + bi$.

      i. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

      ii. $1 + \frac{i}{3} - \frac{1}{9} - \frac{i}{27} + \frac{1}{81} + \dots$

   (d) For what complex numbers z does the geometric series diverge?

2. Reread Theorem 4 (page 204), and study Figure 5.2. Suppose the series $\sum a_k(z-i)^k$ has a radius of convergence $R = 1$.

   (a) Draw a picture of the disc of convergence for this series, and label the regions of (absolute) convergence and divergence.

   (b) For each point, state whether the series (a) certainly converges absolutely, (b) certainly diverges, or (c) possibly converges and possibly diverges at that point.

      i. $z = -i$

      ii. $z = \frac{i}{2}$

      iii. $z = 0$

3. What convergence test (from Calc 2) is the key to computing the radius of convergence for a complex power series?

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____

Read Section 5.3, Analytic Functions Yield Power Series.

Reading Questions

1. Look back at Corollary 11 in Section 5.2. What does this corollary say?

2. In Paragraphs 5.3.1 and 5.3.2, we work on proving a converse to this corollary, namely (in Theorem 15) we show that an analytic function has a local power series representation (the Taylor series). State Theorem 15 here.

3. List the four methods for finding Taylor series given in Paragraph 5.3.3.

4. There are a couple very interesting applications presented in Paragraphs 5.3.4 and 5.3.5.

   (a) Theorem 16 in Paragraph 5.3.4 is sometimes called "The Isolated Zeros Theorem". Suppose an analytic function f has a zero at z_0, i.e. $f(z_0) = 0$. There are two possibilities for the local behavior of f near z_0. What are they?

   (b) Theorem 18 in Paragraph 5.3.5 is sometimes called "The Identity Principle." It can be used to show that if two analytic functions on a domain Ω agree on a set S with an accumulation point in Ω, then the two functions must be the same throughout Ω. In particular this implies the uniqueness of analytic extensions of functions on $\mathbb{R}$. (For example, although we might imagine that there are many possible analytic extensions of e^x to the complex plane, there is, in fact, only one!) *This is not really a question. Just a comment.*

5. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 6.1, The Three Types of Isolated Singularity.

Reading Questions

1. What is an isolated singularity?

2. List the three types of isolated singularities, and give an example of each.

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____________________________________

Read Section 6.2, Laurent Series.

Reading Questions

1. What makes a Laurent series different from a Taylor series?

2. What is the shape of the domain of convergence of a Laurent series?

3. Theorem 2 gives an integral formula for the coefficients of a Laurent series.

   (a) Look over the examples in Paragraph 6.2.2. Do we use the integral formula in any of these examples?

   (b) Reread the third comment after Theorem 2. In your own words, what is this comment saying?

   (c) Use the Laurent series for $e^{1/z}$ from Example 3 in Paragraph 6.2.2 to find the value of the integral:

   $$\frac{1}{2\pi i}\int_{C(0,1)} e^{1/z}\,dz$$

   (Hint: According to the integral formula in Theorem 2, this integral represents the coefficient, c_{-1}, for z^{-1} in the Laurent series for $e^{1/z}$.)

4. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 6.3, Poles.

Reading Questions

1. Theorem 4 gives a classification of isolated singularities by Laurent coefficients. Suppose f has an isolated singularity at z_0 and f. Three possible forms of the Laurent series are given below. For each case, say what type of singularity f has at z_0.

   (a) $c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$

   (b) $\dfrac{c_{-m}}{(z - z_0)^m} + \ldots + \dfrac{c_{-2}}{(z - z_0)^2} + \dfrac{c_{-1}}{(z - z_0)} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$

   (c) $\ldots + \dfrac{c_{-2}}{(z - z_0)^2} + \dfrac{c_{-1}}{(z - z_0)} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots$

2. Reread the "Further Discussion" near the end of the section.

   (a) For the function below, list the zeros and poles, and state the orders of the zeros and poles.

$$f(z) = \frac{(z-1)^2}{(z-2i)^3}$$

(b) For the function below, list the poles, and state the orders of the poles.

$$f(z) = \frac{e^z}{z(z-1)^2}$$

3. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ______________________________

Read Section 6.4, Essential Singularities.

Reading Questions

1. Reread the *Illustration* part of the section. Suppose D' is the punctured disc around the origin of radius $\frac{1}{2}$, i.e. the set consisting of all z satisfying $0 < |z| < \frac{1}{2}$.
   (a) Sketch D' and its image under the mapping $z \mapsto \frac{1}{z}$.

   (b) Choose a specific horizontal strip of height 2π that is contained in the image described above, and sketch it.

   (c) What is the image of this horizontal strip under the complex exponential map? Sketch it. (This is the image of D' under the map $z \mapsto e^{1/z}$.)

   (d) Explain how you know that, no matter how small we shrink D', as long as it is a disc of positive radius, its image under $z \mapsto e^{1/z}$ will be $\mathbb{C} - \{0\}$.

2. What struck you in this reading? What is still unclear? What remaining questions do you have?

Name: ____________________

Read Section 7.1, The Residue Theorem.

Reading Questions

1. Reread the first two pages of Section 7.1.

   (a) What is the Laurent series for $e^{1/z}$? (See Example 3 in Paragraph 6.2.2.)

   (b) What is the residue of $e^{1/z}$ at the origin?

   (c) What is the value of the integral $\int_{C(0,1)} e^{1/z}\, dz$?

2. Reread Paragraph 7.1.2.

   (a) What is the formula for computing the residue at a simple pole? (This is near the top of page 267.)

   (b) What is the formula for computing the residue at a double pole? (This is near the bottom of page 267.)

3. Consider the function $f(z) = \dfrac{z+1}{z(z-i)^2}$ which is discussed (briefly) starting at the bottom of page 267.) Use the formulas for computing residues to find the residues of f at $z = 0$ and $z = i$.

4. What struck you in this reading? What is still unclear? What remaining questions do you have?