**Math 1151, Chebyshev Polynomials**
Amy DeCelles
March 11, 2010

To get the Chebyshev polynomial for $\sin n\theta$ or $\cos n\theta$:

1. If $n$ is odd, use the sum formula: e.g. $\sin 3\theta = \sin(\theta + 2\theta) = \dots$
2. Then you will have some $\sin m\theta$'s and $\cos m\theta$'s, where now, $m$ is even.
3. Use the double angle formulas: e.g. $\sin 4\theta = \sin(2 \cdot 2\theta) = \dots$
4. Repeat until all you have are powers of $\sin\theta$ and $\cos\theta$.
5. Use the pythagorean identity to eliminate either the sines or the cosines, so that what you have left a polynomial $\cos\theta$ or $\sin\theta$.

For example:

$$\begin{aligned}
\cos 5\theta &= \cos(\theta + 4\theta) \\
&= \cos\theta\cos 4\theta - \sin\theta\sin 4\theta \qquad \text{(sum formula)} \\
&= \cos\theta\big(2\cos^2 2\theta - 1\big) - \sin\theta\big(2\sin 2\theta\cos 2\theta\big) \qquad \text{(double-angle formulas)} \\
&= 2\cos\theta\cos^2 2\theta - \cos\theta - 2\sin\theta\sin 2\theta\cos 2\theta \\
&= 2\cos\theta\big(2\cos^2\theta - 1\big)^2 - \cos\theta - 2\sin\theta\big(2\sin\theta\cos\theta\big)\big(2\cos^2\theta - 1\big) \qquad \text{(double-angle formulas)} \\
&= 2\cos\theta\big(4\cos^4\theta - 4\cos^2\theta + 1\big) - \cos\theta - 4\sin^2\theta\cos\theta\big(2\cos^2\theta - 1\big) \\
&= (8\cos^5\theta - 8\cos^3\theta + 2\cos\theta) - \cos\theta - 4\sin^2\theta\cos\theta\big(2\cos^2\theta - 1\big) \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta(1 - \cos^2\theta)(2\cos^2\theta - 1) \qquad \text{(pythagorean identity)} \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta(-2\cos^4\theta + 3\cos^2\theta - 1) \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta + 8\cos^5\theta - 12\cos^3\theta + 4\cos\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$